Math 549 – HW 6 Solutions

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Problem 6-2

Let $P_v : \mathbb{R}^{2n+1} \to v^\perp$ denote the orthogonal projection. As discussed in class, $P_v|_M$ is an immersion if and only if $v \notin T_pM$ for any $p \in M$. This is equivalent to $v \notin \text{im}(G)$. Now since $\dim(UM) = 2n-1$ and $\dim(\mathbb{RP}^{2n}) = 2n$, by Sard's theorem, the image of G is null, so there exist v outside the image of G. □

Problem 6-7

Suppose $F_s, 0 \leq s \leq 1$ is a homotopy rel A from $F = F_0$ to a smooth map F_1. Then $F_1(t) = F(t) = (t, |t|)$ for $t \geq 0$. By Taylor expansion, the second component of F is given by $t + O(t^2)$ for $|t| \ll 1$. In particular it is negative for t small but negative, so that F_1 is not valued in $\mathbb{H}^2$. This is a contradiction. □

Problem 6-10

Since F restricts to a map $F|_W : W \to X$, it is evident that $DF(TW) \subseteq TX$, so $TW \subseteq DF^{-1}(TX)$. It remains to establish the reverse inclusion, which is done by dimension count: For every manifold, let the corresponding lower case letter denote its dimension. Also write $k := \dim \ker D_pF$ and $r := \dim \text{im} D_pF$.

We know $n - w = m - x$ (see Corollary 6.31) so $w = n + x - m$. We also know that the intersection $\text{im}D_pF \cap T_{F(p)}X$ has dimension $x+i-m$, so $(D_pF)^{-1}(T_{F(p)}X)$ has dimension $x+i-m+k = x+i-m+(n-i) = x+n-m = w$, where we used rank-nullity in the first identity. □

Problem 6-13(a)

We will use the identifications $T(N \times N') = TN \times TN'$ and $T(M \times M) = TM \times TM$. Note that $T\Delta(M) = \Delta TM \subseteq TM \times TM$. Let $x \in N$ and $x' \in N'$ such that $F(x) = F(x')$ and write y for this point.

Suppose first $F \pitchfork F'$. Let $(v, w) \in T_yM \times T_yM$. We need to find $u \in T_xN$ and $u' \in T_{x'}N'$ such that $(v, w) - (DF(u), DF'(u')) \in T\Delta M$, i.e. such that

$$v - DF(u) = w - DF'(u').$$

By transversality, there exist u, u' such that $DF(u) - DF(u') = w - v$, as desired.

Conversely, suppose $(F \times F') \pitchfork \Delta M$ and let $v \in T_yM$. Choose $u \in T_xN$ and $u' \in T_{x'}N'$ and $w \in T_yM$ such that

$$(v, 0) = (DF(u) + w, DF(u') + w).$$

From the second component we see that $w = -DF'(u')$. Then the first component shows $v = DF(u) - DF'(u')$, as desired. □

Problem 6-16(d)

We assume M, and hence also N, is connnected (the general case is straightforward by considering connected components of M). Consider the map $F : N \times S \to M$. At every point of the form (p, s_0), the derivative of the left-hand minor of the matrix (corresponding to the coordinates on N in a product chart for $N \times S$) is invertible at and hence each such point has a small open neighborhood $U_p \times V_p$ on which the derivative is also invertible. Since N is compact, we can pass to a finite subcover $\{U_{p_i}\}$, and set $V := \cap_i V_{p_i}$. Then for $s \in V$ and $p \in N$, we know that $(p, s) \in U_{p_i} \times V$ for some i, and hence the derivative of F_s is invertible at p. We conclude that for each $s \in V$, the map F_s is a local diffeomorphism.

Since N is compact, any local diffeomorphism is surjective (because by compactness, the image is closed; on the other hand, the image under any local diffeomorphism is open; since M is connected, the image must then be all of N). Therefore it remains to show F_s is injective.

Injectivity, proof using covering space theory: Since N is compact, any map from N is automatically proper. A proper local diffeomorphism is a covering map. By the classification of covering spaces, a covering map is a diffeomorphism if and only if it induces an isomorphism on fundamental groups. In particular, F_{s_0} induces an isomorphism. For s sufficiently close to s_0 (say contained in a chart of S around s_0), the maps F_s and F_{s_0} are homotopic, so F_s also induces an isomorphism on fundamental groups.

Injectivity, proof without covering space theory: We argue by contradiction so suppose there exists a sequence $s_n \to s_0$ and distinct pairs of points $x_n, y_n \in N$ such that $F(x_n, s_n) = F(y_n, s_n)$. Since N is compact, by passing to subsequences we can assume that $x_n \to x_0$ and $y_n \to y_0$. Since F_{s_0} is injective, we must have $x_0 = y_0$.

On the other hand, F is a submersion, so $Z := F^{-1}(x_0)$ is an embedded submanifold of $N \times S$. The projection to the second factor $F^{-1}(x_0) \to S$ is a local diffeomorphism (because its derivative is an isomorphism). Therefore on a neighborhood of s_0, it is actually a diffeomorphism. This means that for every s sufficiently close to s_0, there is a unique point $z(s) \in F^{-1}(x_0)$. This contradicts the existence of x_n and y_n above. □