Math 549 – HW 7 Solutions

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Problem 6-5

Proof of the hint: Let $y_0 \in \mathbb{R}^n$ and $x_0 \in M$ that minimizes $\|x_0 - y_0\|$. Set $f(x) := \|x - y_0\|^2$. Note $D_{x_0} f = 2\langle \cdot, x_0 - y_0 \rangle$. As x_0 is a local minimum of f on M, we must have $D_{x_0} f = 0$ on $T_{x_0}M$, i.e. $x_0 - y_0 \perp T_{x_0}M$, as desired.

Now we prove that for every x, there exists $\delta > 0$ such that if $\|y - x\| < \delta$, then x is the unique closest point to y on M. Near x, you can write M as the graph of a function $F : T_xM \to (T_xM)^\perp$ with $D_0F = 0$. Let $v \in T_xM$ and write $c(t) = (tv, F(tv)) \in M$. Then $c(0) = x$ and we have

$$\langle c(t) - y, \pm c'(t) \rangle = t(\|c'(0)\|^2 \pm \langle x - y, c''(0) \rangle) + O(t^2).$$

By choosing the sign appropriately, we can arrange for the first order term to be nonzero. This shows $c(t) - y$ is not perpendicular to $c'(t) \in T_{c(t)}M$ for $|t| \ll 1$. In fact the bound on t only depends on F, so that there exists $\delta > 0$ such that for all v and $|t| < \delta$, we have that $y - c(t)$ is not perpendicular to $T_{c(t)}M$. ☐

Problem 6-9

Fix $r > 0$ and let $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0, y_0) \in S_r(0)$. Note that $\|F(x, y_0)\| = r$ for any x, so $F_*\partial_x \in TS_r(0)$. Therefore $T_{F(x_0,y_0)}S_r(0) + \mathrm{im} D_{(x_0,y_0)}F = \mathbb{R}^3$ if and only if $F_*\partial_y$ has nonzero radial component, i.e. $\langle F_*\partial_y, (x_0, y_0) \rangle \neq 0$. We compute

$$\langle F_*\partial_y, (x_0, y_0) \rangle = e^{2y_0} - e^{-2y_0}$$

which is nonzero precisely when $y_0 \neq 0$. On the other hand, the points $F(x, 0) \in S_1(0)$ for all x, so $F \pitchfork S_r(0)$ if and only if $r \neq 1$.

Problem 7-2

Note that on each factor $G \times \{e\}$ and $\{e\} \times G$ (identified with G in the obvious way), the map m restricts to the identity map, so its derivative is the identity. The result then follows by linearity of the derivative and $T(G \times G) = TG \times TG$. ☐

Problem 7-4

(a) Suppose first that A is diagonalizable eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any $t \in \mathbb{R}$, note $I + tA$ is diagonalizable with eigenvalues $1 + t\lambda_i, 1 \leq i \leq n$. Therefore (using binomial formula)

$$\det(I + tA) = \prod_i (1 + t\lambda_i) = 1 + t\sum_i \lambda_i + O(t^2) = 1 + t\mathrm{tr}(A) + O(t^2).$$

(Alternatively, as the hint suggests, you can deduce this from B.3 for general A.)

So $\frac{d}{dt}|_{t=0} \det(I + tA) = \mathrm{tr}(A)$. Since both sides of this equation are continuous in A and diagonalizable matrices are dense, the identity in fact holds for all A (you already knew this if you used B.3 instead), as desired. You could also use existence of a basis for $\mathfrak{gl}(n, \mathbb{R})$ consisting of diagonalizable matrices, and use the linearity of both trace and the derivative of determinant. ☐

(b) The idea is to use that the derivative of det at identity is computed in (a), left-translation l_X by X is a diffeomorphism taking identity to X that is linear on the space of matrices (so its derivative is again left-translation by X), and finally that det is a homomorphism. More precisely we have so

$$\begin{aligned} D_X \det(B) &= D_X \det(l_X(X^{-1}B)) && (B = l_X(X^{-1}B)) \\ &= D_e(\det \circ l_X)(X^{-1}B) && \text{(chain rule and linearity of } l_X\text{)} \\ &= D_e(l_{\det(X)} \circ \det)(X^{-1}B) && \text{(det is a homomorphism)} \\ &= \det(X)\operatorname{tr}(X^{-1}B) && \text{(chain rule, (a), and linearity of multiplication of real numbers)} \end{aligned}$$

Problem 8-1

Let $U \supseteq A$ be an open neighborhood such that X has a smooth extension (which we will also denote by X) to U, let and $V \subseteq U$ open with closure contained in U. Let f be a bump function supported on U with $f = 1$ on A (the existence of f was proved in a previous homework problem). Then set $\widetilde{X} := fX$ on U and extend by 0 outside U. □

Problem 8-4

It suffices to construct one of them, say an outward pointing vector field, since the other one can then be obtained by multiplying by -1.

Also it suffices to construct such a vector field on ∂M, since it can then be extended to M by virtue of the previous problem (8-1). Recall that ∂M is itself a manifold. Let $\{f_\alpha\}_\alpha$ be a partition of unity subordinate to some atlas of charts $\{U_\alpha\}_\alpha$ on ∂M. On each chart U_α, we have an outward pointing vector field ν_α by pulling back such a vector field from $\partial\mathbb{H}^n$. Then set $\nu := \sum_\alpha f_\alpha \nu_\alpha$. Using convexity of the space of outward pointing vector fields on each U_α, it easily follows that ν is outward pointing on all of ∂M.

Problem 8-16

Using the formula for Lie bracket in coordinates, it is straightforward to compute:

(a) $[X, Y] = 4xy\partial_y - \partial_z$,

(b) $[X, Y] = x\partial_z - z\partial_x$,

(c) $[X, Y] = 2x\partial_x - 2y\partial_y$.

Problem 8-18

(a) Then the derivative is an isomorphism, so given $x \in M$, set $X(x) := (D_xF)^{-1}(Y(F(x)))$. This is smooth because F is a local diffeomorphism, i.e. locally we can write $X = (F^{-1})_*Y$.

(b) Cover M by charts $(U_\alpha, \varphi_\alpha)$ such that F is projection on the first n coordinates in these coordinates. On U_α, we have a lift $X_\alpha(x) = (Y(F(x)), 0, \ldots, 0)$ (written out in these coordinates). Note that it is not unique since we can replace the 0's with any other vector. Since being a lift is invariant under convex combinations, we can patch together such local lifts with a partition of unity to a global lift $X = \sum_\alpha f_\alpha X_\alpha$. It is not unique since changing one of the X_α gives a different lift.

(c) If X is the lift of Y, then $Y(F(p)) = Y(F(q))$ which gives $D_pF(X(p)) = D_qF(X(q))$. Conversely: If Y exists, we must have $Y(F(p)) = D_pF(X(p))$, which proves the uniqueness. Further note the right-hand side is well-defined, and X is indeed a lift of Y. Finally, one may see that Y is smooth by taking local coordinates so that F is a projection map; in such coordinates $D_pF = F$ so Y is just the projection of X.

(d) First suppose X is a lift of Y, and let V be any vertical vector field. Then

$$F_*[V, X] = [F_*V, F_*X] = [0, X] = 0,$$

so $[V, X]$ is vertical.

Conversely, suppose $[V, X]$ is vertical for any vertical vector field V. We attempt to verify the criterion from (c). An easy argument using path-connectedness of fibers of F, shows that without loss of generality p, q belong to a single chart. So we choose a chart where F is given by projection to the first n coordinates, and write $X = \sum_i X^i \partial_i$. Our goal is to show that X^i are independent of x_j for $i \leq n \leq j$. Letting $V = \partial_j$, we have

$$[V, X] = \sum_i (\partial_j X^i) \partial_i.$$

Since $[V, X]$ is vertical (note that we can easily assume that V extends to all of M by 8-1), we have $\partial_j X^i = 0$ for $i \leq n$, as desired. $\square$