Math 549 – HW 8 Solutions

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Problem 8-25

Proof of the hint: Let $g, h \in G$, then we have

$$\begin{aligned} i(gh) &= (gh)^{-1} = h^{-1}g^{-1} \\ &= g^{-1}h^{-1} = i(g)i(h) \end{aligned}$$

where at the start of the second line, we used that G is abelian. In addition $i(e) = e$, so i is a homomorphism. Then $i_* = D_e i$ is a Lie algebra homomorphism of $\mathfrak{g}$. By Problem 7-2, $i_* = -\mathrm{Id}$, so for any $X, Y \in \mathfrak{g}$, we have

$$-[X, Y] = i_*[X, Y] = [i_*X, i_*Y] = [-X, -Y] = [X, Y],$$

so $2[X, Y] = 0$ so $[X, Y] = 0$. □

Problem 9-1(a)

Assume γ is not constant and not injective. We will show γ is periodic. Since γ is not injective, there exist $a < b$ such that $\gamma(a) = \gamma(b)$. Consider the set $P := \{T \in \mathbb{R}^+ \mid \gamma(a) = \gamma(a + T)\}$. Since γ is not constant, $X(\gamma(a)) \neq 0$, hence for $0 < t \ll 1$, we have $\gamma(t) \neq \gamma(a)$ (because you can choose local coordinates near $\gamma(a)$ such that $X = \partial_1$ so that $\gamma(a + t) = te_1 \neq 0$ in these coordinates).

Therefore P is closed (not just in $\mathbb{R}$ but in $\mathbb{R}^+$ as well) and hence contains its infimum, call it T. We will prove that $\gamma(t + T) = \gamma(t)$ for all t. Let Φ be the flow of X and write $p = \gamma(0)$.

Then

$$\begin{aligned} \gamma(t + T) &= \Phi_{(t-a)+a+T}(p) = \Phi_{t-a}\Phi_{a+T}(p) \\ &= \Phi_{t-a}\gamma(a + T) \\ &= \Phi_{t-a}\gamma(a) \\ &= \gamma(t). \end{aligned}$$

□

Problem 9-6

Let K be any compact set. Let $\varepsilon > 0$ such that every integral curve starting in K exists for time at least ε. Since the integral curve starting at $\gamma(b - \delta)$ exists for time less than ε whenever $0 < \delta < \varepsilon$, we see that $\gamma(b - \delta) \notin K$ for any $0 < \delta < \varepsilon$. □

Problem 9-7

Proof of the hint: Just take $\theta = q - p$ on the line segment $[p, q]$, and use the extension lemma for vector fields to extend θ to $\mathbb{B}^n$.

Solution of the problem itself: Let $p \in M$ and let O be the orbit of p under $\mathrm{Diff}(M)$. Note that O is nonempty and open by the hint (the vector fields produced in the hint on a chart around p can be extended to compactly supported vector fields M using the extension lemma, hence their flows also extend; the flows exist for all time because compactly supported vector fields are complete).

O is also closed, because whenever a group action has open orbits, the orbits are also closed (the complement of an orbit is a union of (open) orbits, therefore the complement is open).

Since M is connected and O is nonempty clopen, we must have $O = M$. □

(You can also solve this problem by taking a path from p to q and "flowing along the path". You cover the path by charts, and compose the diffeomorphisms produced by the hint. You have to extend the vector fields to the ambient manifold and argue completeness just as in the above solution.)

Problem 9-16

One may find such vector fields by first taking $V = f(x, y)\partial_x$ and $W = \partial_x + h(x, y)\partial_y$. We see that we need $f(x, 0) = 1$ and $h(x, 0) = 0$. It will further be convenient to require f and h to be functions of y only, so that $[V, W]$ does not get contributions from the ∂_x-derivatives, and we have

$$[V, W] = -h(y)\partial_y f \partial_x.$$

For example, taking $V = (1 + y)\partial_x$ and $W = \partial_x + y\partial_y$, we find $[V, W] = -y\partial_x \neq 0$. □

Problem 9-17

We need to check for linear independence at $(1, 0, 0)$ and determine whether all Lie brackets vanish.

(a) The vector fields are clearly linearly independent, and commute since coordinate vector fields commute. Explicit coordinates can be found by (linear) change of basis.

(b) $V_1(1, 0, 0) = V_2(1, 0, 0)$ so they are not linearly independent.

(c) Since $V_2(1, 0, 0) = 0$, these vector fields are not a local frame near $(1, 0, 0)$, and in particular not a local coordinate frame.