Math 549 – HW 9 Solutions

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Problem 10-6

Define an equivalence relation on $\sqcup_{\alpha\in A}(U_\alpha \times \mathbb{R}^k)$ where $(x,v) \sim (y,w)$ with $(x,v) \in U_\alpha \times \mathbb{R}^k$ and $w \in U_\beta \times \mathbb{R}^k$ if and only if $x = y$ and $w = \tau_{\alpha\beta}(x)v$. One checks this is an equivalence relation ($\tau_{\alpha\alpha} = \mathrm{Id}$ shows reflexivity; symmetry holds because $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$ and transitivity follows from the cocycle identity).

Set $E := (\sqcup_\alpha(U_\alpha \times \mathbb{R}^k))/\sim$ and define $\pi : E \to M$ via $\pi([x,v]) = x$. Note that this is well-defined.

Topologize E with the quotient topology. For each α, the quotient map restricted to $U_\alpha \times \mathbb{R}^k$ is injective (to E). The inverse of such a map is the local trivialization Φ_α. Using the definition of the equivalence relation, it is easy to verify $\Phi_\beta \circ \Phi_\alpha^{-1} = \tau_{\alpha\beta}$. It follows that if $V \subseteq U_\alpha \times \mathbb{R}^k$ is open, then the image of V in E is open. It easily follows that E is Hausdorff and locally Euclidean. To show it is second countable, you can use a collection of the form $\{V_i \times W_j\}_{i,j}$, where $\{V_i\}_i$ is a countable base of M with each element contained in some U_α, and $\{W_j\}_j$ a countable base of $\mathbb{R}^k$. It is straightforward to check this is a base of E.

Charts on E can be defined by taking $V \subseteq U_\alpha$ to be a chart $\varphi : V \to \mathbb{R}^n$, and then considering $\varphi \times \mathrm{id} : V \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$. The transition functions are given by maps of the form $(\psi \circ \varphi^{-1}) \times \tau_{\alpha\beta}$, and are therefore smooth. So E is a smooth manifold.

On one of the above charts $V \times \mathbb{R}^k$, the map $\pi : E \to M$ is projection to the first coordinate and therefore smooth. ☐

Problem 10-10

Let $U_1, \ldots, U_N$ be a finite cover of M by local trivializations, and let $\{f_i\}_{1\leq i\leq N}$ be a partition of unity subordinate to this open cover. Define a map

$$F : M \times \mathbb{R}^{kN} \to E, (x, v_1, \ldots, v_N) \mapsto (x, \sum_i f_i(x)v_i)$$

where, if $x \notin U_i$, we interpret $f_i(x)v_i = 0$ as the zero vector in $\pi^{-1}(x)$. Then f is a submersion: Given $x \in M$, choose i with $f_i(x) \neq 0$. Then identifying $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^k$, the restriction of F to $U_i \times \mathbb{R}^k$ with respect to these coordinates on E is $(x,v) \mapsto (x, f_i(x)v)$, which has derivative given by a block matrix with identity in the top left corner and $f_i(x)\mathrm{Id}$ in the bottom right corner, and hence is surjective.

By the parametric transversality theorem, for almost every $v \in \mathbb{R}^{kN}$, the map $\sigma := F_v : M \to E$ is transverse to the zero section Z. Pre-images of submanifolds under maps transverse to them are embedded submanifolds, so if $k \leq \dim(M)$, then $\sigma^{-1}(Z) = \{x \in M \mid \sigma(x) = 0\}$ is a closed (hence compact) embedded submanifold of M.

Finally suppose $k > \dim(M)$ and $\sigma(x) = 0$. Then $\mathrm{im} D_x\sigma + T_{(x,0)}Z = T_{(x,0)}E$. Since $\dim(E) = \dim(M) + k$ and $\dim(Z) = \dim(M)$, it follows that $D_x\sigma$ has rank at least k. But this is impossible since $k > \dim(M)$. Therefore there does not exist any point $x \in M$ with $\sigma(x) = 0$, i.e. σ is nowhere vanishing. ☐

Problem 10-15

Let $S_0 \in \mathrm{Gr}_k(V)$. Let $U \ni S_0$ be the open neighborhood consisting of those subspaces $S \subseteq V$ such that $S \cap S_0^\perp = 0$. Let $v_1, \ldots, v_k \in S_0$ be a basis and write $\sigma_i(S) := \mathrm{proj}_S(v_i)$. It is straightforward to check that $\sigma_1(S), \ldots, \sigma_k(S)$ are a basis of S (because any linear relation gives a linear combination of $v_1, \ldots, v_k$ that belongs to $S^\perp$). The map

$$U \times \mathbb{R}^k \to \pi^{-1}(U), (S, a_1, \dots, a_k) \mapsto \sum_i a_i \sigma_i(S)$$

is then a local trivialization of T over U. $\square$

Problem 10-17

Note that $TM \oplus NM = \{(x, v, w) \mid v \in T_xM \text{ and } w \in N_xM\}$. Consider then the map

$$F : TM \oplus NM \to T\mathbb{R}^n|_M, (x, v, w) \mapsto (x, v + w).$$

This is a bundle homomorphism since it is linear on fibers and lifts identity on M. Further it is a linear isomorphism on each fiber. Therefore F is a bundle isomorphism.

Problem 11-5

We will prove that for any vector bundle, E is trivial if and only if E^* is trivial. If $E = M \times \mathbb{R}^k$ is trivial, then $E^* = M \times (\mathbb{R}^k)^* \cong M \times \mathbb{R}^k$ is also trivial. Conversely, note that if E^* is trivial, then E^{**} is trivial (by the above argument). But $E^{**} \cong E$ since they have the same cocycle. $\square$

Problem 11-7(a)

For any $(s, t) \in \mathbb{R}^2$ and $v \in T_{(s,t)}\mathbb{R}^2 \cong \mathbb{R}^2$, we have

$$(F^*\omega)_{(s,t)}v = \omega_{F(s,t)}(F_*v) = st dy(F_*v) - e^t dx(F_*v).$$

Writing $v = (v_1, v_2)$, we have $F_*v = (tv_1 + sv_2, e^t v_2)$. Now $dx(w_1, w_2) = w_1$ and similarly for dy, so we have

$$\begin{aligned}(F^*\omega)_{(s,t)}v &= ste^t v_2 - e^t(tv_1 + sv_2) \\ &= -te^t ds(v) + se^t(t-1)dt(v),\end{aligned}$$

so $F^*\omega = -te^t ds + se^t(t-1)dt$.

Problem 11-14

(a) Set $c(t) = (t, t, t)$. We have

$$\int_c \omega = \int_0^1 \omega_{c(t)}(c'(t))dt.$$

Using $c'(t) = (1, 1, 1)$, we find

$$\omega_{c(t)}(c'(t)) = -\frac{4t}{(t^2+1)^2} + \frac{4t}{t^2+1}$$

so

$$\begin{aligned}\int_c \omega &= \int_0^1 \left(-\frac{4t}{(t^2+1)^2} + \frac{4t}{t^2+1}\right) dt \\ &= \left[\frac{2}{t^2+1} + 2\log(t^2+1)\right]_{t=0}^1 \\ &= 2\log(2) - 1.\end{aligned}$$

Similarly,

$$\eta_{c(t)}(c'(t)) = -\frac{4t^2}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2}{t^2+1}.$$

Integrating the above, we find

$$\int_c \eta = \left[\frac{2t}{t^2+1} + \log(t^2+1)\right]_{t=0}^{1} = 1 + \log(2).$$

(b) Suppose $\omega = df = f_x dx + f_y dy + f_z dz$. Using $f_z = 2x/x^2+1$, we see $f = 2xz/(x^2+1) + g(x,y)$ for some function g. Then $f_x = 2z/(x^2+1) - 4x^2z/(x^2+1)^2 + g_x(x,y)$, but this cannot be equal to $-4z/(x^2+1)^2$, so ω is not exact.

Now suppose $\eta = df$. In the same manner, we find the solution $f(x,y,z) = 2z/(x^2+1) + \log(y^2+1)$.

(c) $\int_c \eta = f(1,1,1) - f(0,0,0) = 1 + \log(2)$.