MUTUAL STATIONARITY, STATIONARY REFLECTION, AND THE FAILURE OF SCH

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Abstract. We give another proof that from large cardinals the failure of SCH at $\aleph_\omega$ is consistent with mutual stationarity at $\langle \aleph_n \cap \mathrm{cof}(\omega_k) \mid k < n < \omega \rangle$ for all $k < \omega$. This new proof uses extender-based forcing rather than standard Prikry forcing. We then show that it is consistent for these properties to hold along with stationary reflection at $\aleph_{\omega+1}$.

1. Introduction

A subset S of κ is *stationary* if S intersects every closed and unbounded subset of κ. Stationary sets have an equivalent model-theoretic definition: given a regular cardinal κ and a cardinal $\lambda > \kappa$, a set $S \subseteq \kappa$ is stationary if and only if for every algebra $\mathfrak{A}$ on λ, there is an elementary submodel $N \prec \mathfrak{A}$ such that $\sup(N \cap \kappa) \in S$.

This characterization was used by Foreman and Magidor [6] to define *mutual stationarity.* Given a sequence of regular cardinals $\langle \kappa_n \mid n < \omega \rangle$ with limit κ, a sequence $S_n \subseteq \kappa_n$ of stationary sets is mutually stationary if for every algebra on κ, there is a single elementary substructure N that simultaneously witnesses the stationary of each set. In full generality, we have the following definition:

Definition 1.1. Let R be a set of uncountable regular cardinals, and let $\vec{S} = \langle S_\kappa \mid \kappa \in R \rangle$ be a sequence of stationary sets with $S_\kappa \subseteq \kappa$. The sequence $\vec{S}$ is *mutually stationary* if for every algebra $\mathfrak{A}$ on $\sup(R)$, there is $N \prec \mathfrak{A}$ such that $\sup(N \cap \kappa) \in S_\kappa$ for all $\kappa \in R \cap N$.

If R is a cofinal sequence in some singular cardinal κ, the statement that every sequence of stationary sets $\langle S_\kappa \mid \kappa \in R \rangle$ is mutually stationary can be viewed as property of κ.

It is sometimes useful to only consider certain stationary sets. Suppose $R = \langle \kappa_n \mid n < \omega \rangle$, with limit κ, and let $A_n \subseteq \kappa_n$ for each $n < \omega$. We say that *mutual stationarity holds at* $\langle A_n \mid n < \omega \rangle$ if every sequence of stationary sets $S_n \subseteq A_n$ is mutually stationary. In particular, we might often restrict to stationary sets of certain fixed cofinality, setting each A_n to be $\kappa_n \cap \mathrm{cof}(\tau)$ for some regular τ.

Restricting to stationary sets of countable cofinality, Foreman and Magidor [6] showed that mutual stationarity always holds at $\langle \kappa_n \cap \mathrm{cof}(\omega) \mid n < \omega \rangle$. This behavior does not generalize to larger fixed cofinalities; they showed that in L, there is a sequence of stationary sets $S_n \subseteq \aleph_n \cap \mathrm{cof}(\omega_1)$ for $n > 1$ that is not mutually stationary. This sparked the following question: is it consistent for mutual stationarity to hold at the $\aleph_n$'s, restricted to fixed cofinality greater than $\aleph_0$?

In the past few decades, there have been a number of partial results. Cummings, Foreman, and Magidor [5] showed that in a generic extension for Prikry forcing to singularize a cardinal κ, if $\langle \kappa_n \mid n < \omega \rangle$ is the Prikry sequence then every sequence of stationary sets $S_n \subseteq \kappa_n$ is mutually stationary. Koepke [9] forced mutual stationarity for $\langle \aleph_{2n+1} \cap \mathrm{cof}(\omega_1) \mid 1 < n < \omega \rangle$ from a measurable cardinal, and Koepke and Welch [10] showed that a measurable cardinal is necessary to have mutual stationarity for $\langle \kappa_n \cap \mathrm{cof}(\omega_1) \mid n < \omega \rangle$.

Recently the question was answered by Ben-Neria [2], who showed that from countably many supercompacts it is consistent that every sequence of stationary sets $S_n \subseteq \aleph_n$ of some fixed cofinality is mutually stationary. His model uses an iteration of Levy collapses, forcing to make the supercompacts become the $\aleph_n$'s. In this model, the singular cardinal hypothesis holds at $\aleph_\omega$.

In [1], Sinapova and the author showed that it was consistent from three supercompact cardinals for SCH to fail at $\aleph_\omega$ while mutual stationarity holds at $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid n > k \rangle$ for all $k < \omega$. The forcing used was a Prikry forcing with interleaved collapses, singularizing a supercompact cardinal and then using the reflection along the Prikry sequence to obtain mutual stationarity. In this model there are many failures of GCH below $\aleph_\omega$.

Another property stemming from stationary sets is stationary reflection. A stationary set $S \subseteq \kappa$ *reflects at* α if $S \cap \alpha$ is stationary in α; we say that S *reflects* if S reflects at some α. We say that stationary reflection holds at κ if every stationary subset of κ reflects.

From countably many supercompacts, Magidor [11] forced stationary reflection to hold at $\aleph_{\omega+1}$. SCH holds in this construction. Combining stationary reflection with the failure of SCH has been the subject of much recent work. Ben-Neria, Hayut, and Unger [3] and Poveda, Rinot, and Sinapova [12] independently proved that it is consistent for SCH to fail at a singular strong limit cardinal κ with the stationary reflection at κ^+. Poveda, Rinot, and Sinapova [13] added collapses to their construction, showing that stationary reflection at $\aleph_{\omega+1}$ and and the failure of SCH at $\aleph_\omega$ could consistently coexist.

Their construction is a complicated and intricate iteration scheme, which uses extender-based forcing as its starting point. Extender based forcing obtains the failure of SCH by adding many new cofinal sequences to a singular cardinal rather than singularizing a regular cardinal with large powerset. Collapses can be interleaved into extender-based forcings, as described by Gitik in [8], and these techniques can be used to force properties at $\aleph_\omega$ that may be difficult or impossible to obtain by singularizing a cardinal. The construction of [13] uses extender-based forcing with collapses to violate SCH, and then iterates in a Prikry-type way to destroy all nonreflecting stationary sets.

In this paper, motivated by that construction, we give another proof that the failure of SCH and mutual stationarity for a fixed cofinality can consistently hold at $\aleph_\omega$, using extender-based forcing with interleaved collapses. While this requires a stronger large cardinal hypothesis, countably many supercompacts rather than only three, the failure of SCH is in some sense a stronger example of incompactness than in the model of [1]. In that model, GCH failed at every third cardinal below $\aleph_\omega$, while in the model in this paper GCH will hold below $\aleph_\omega$.

Finally we will combine these techniques with the construction of [13] to obtain the same mutual stationarity result, along with stationary reflection at $\aleph_{\omega+1}$ and the failure of SCH at $\aleph_w$. In particular, we prove the following theorem:

Theorem 1.2. *Suppose there is a sequence of indestructibly supercompact cardinals of length $\omega + 2$. Then there is a forcing extension in which the following hold:*

(1) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$
(3) Stationary reflection holds at $\aleph_{w+1}$
(4) Mutual stationarity holds at $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega\rangle$ *for all* $k < \omega$.

In Section 2, we will discuss the basic techniques we will use to obtain mutual stationarity. These techniques were developed by Ben-Neria [2]; they provide a way to obtain mutual stationarity from the existence of ideals with certain properties. In Section 3, we define Gitik's extender-based forcing with interleaved collapses. Section 4 uses this forcing to give a novel proof that SCH can fail at $\aleph_\omega$ while mutual stationarity holds at the $\aleph_n$'s for any fixed cofinality. Finally, in Section 5 we prove the main theorem.

2. Preliminaries

In this section we summarize techniques due to Ben-Neria [2] we will use throughout this paper to prove mutual stationarity. The key technique in Ben-Neria's argument was using the existence of closed nonstationary ideals to construct the elementary submodels needed for mutual stationarity. We give an overview of these methods; for more details see [2, Section 2] or [1, Section 2].

Definition 2.1. Let R be a set of uncountable regular cardinals and $S = \langle S_\kappa \mid \kappa \in R\rangle$ be a sequence of stationary sets with $S_\kappa \subseteq \kappa$. The sequence S is mutually stationary if for every algebra $\mathfrak{A}$ on $\sup(R)$ there is $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa) \in S_\kappa$ for every $\kappa \in R \cap M$.

Definition 2.2. Suppose that R is an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega\rangle$ with limit κ. Given $A_n \subset \kappa_n$, we will say that mutual stationarity holds at $\langle A_n \mid i < n < \omega\rangle$ if every sequence of stationary sets $S_n \subset A_n$ is mutual stationary.

Our goal is to construct the witness model inductively. Given some model N_k witnessing that S_k is a stationary set, we wish to build an extension N_{k+1} that witnesses the stationarity of S_{k+1}, while still witnessing the stationarity of S_k. The following definition captures the desired properties:

Definition 2.3. Suppose $M \prec \mathfrak{A}$. We call an extension N of M an *end-extension of* M *above* λ if $M \prec N \prec \mathfrak{A}$ such that $N \cap \lambda = M \cap \lambda$.

By the following fact, it is enough to verify mutual stationarity on a tail, so we can start this process at any finite stage n.

Fact 2.4. *[6, Lemma 23] Let* ν *be a regular cardinal less than the least element of a set of regular cardinals* K. *If* $\{S_\kappa \mid \kappa \in K\}$ *is mutually stationary, and for all* κ, $S_\kappa \subseteq \mathrm{cof}(\leq \nu)$, *then for all* $\lambda_1, \ldots, \lambda_n$ *greater than* ν *and not in* K, *and all sequences of stationary sets* $S_{\lambda_i} \subseteq \lambda_i \cap \mathrm{cof}(\leq \nu)$, *the sequence* $\{S_\kappa \mid \kappa \in K\} \cup \{S_{\lambda_i}, \ldots S_{\lambda_n}\}$ *is mutually stationary.*

These end-extensions will be constructed using ideals on κ.

Definition 2.5. A nonprincipal κ-complete ideal I on κ is μ-closed if I^+ has a $\leq_I$-dense subset D such that the restriction $\leq_I \restriction D$ is μ-closed. An ideal on κ is *nonstationary* if it extends the nonstationary ideal.

Lemma 2.6. *[2, Proposition 2.12 and Remark 2.9] Suppose $\mu < \kappa$ are regular cardinals and $\mathfrak{A}$ is an algebra extending $\langle H_\theta, \in, <_\theta \rangle$ for some regular cardinal $\theta > 2^\kappa$. Let $M \prec \mathfrak{A}$ be a substructure of size μ closed under sequences of size $< \mu$, and let $S \subseteq \kappa \cap \operatorname{cof}(\mu)$ be a stationary subset of κ in M. Suppose also that at least one of the following holds:*

(1) S consists of approachable points
(2) κ is inaccessible
(3) $\kappa = \tau^+$ and $\tau^{<\tau} = \tau$.

If S is positive with respect to some nonstationary κ-complete $(\mu+1)$-closed ideal on κ, then for every regular cardinal $\lambda \in M \cap \kappa$, there is a μ-closed substructure $N \prec \mathfrak{A}$ of size μ which is an end-extension of M above λ and satisfies $\sup(N \cap \kappa) \in S$.

To show that suitable end-extensions exist, it suffices to verify the hypotheses of Lemma 2.6 at each stage of the induction. To simplify this process, as in [1] we define a principle which captures the key hypothesis of Lemma 2.6.

Definition 2.7. Let $\nu < \theta$ be uncountable cardinals. We say $\dagger^\nu_\theta$ holds if for all stationary $S \subseteq \theta$, there is a nonstationary θ-complete, $(\nu+1)$-closed ideal, for which S is a positive set. Given a poset $\mathbb{Q}$, we say that $\dagger_{\theta,\mathbb{Q}}$ holds if $1_\mathbb{Q}$ forces that for all uncountable ν with $\nu^{++} < \theta$, for all stationary $\dot{S} \subseteq \theta$, there is a nonstationary θ-complete, $(\nu+1)$-closed ideal, for which $\dot{S}$ is a positive set.

By the previous lemma, to ensure that mutual stationarity holds below $\aleph_\omega$ for sets of points of cofinality $\aleph_k$, it suffices to check that $\dagger^{\aleph_k}_{\aleph_n}$ holds for cofinitely many $n < \omega$ and that all relevant stationary sets are approachable. More precisely:

Corollary 2.8. *Suppose that for some $k < \omega$, $\dagger^{\aleph_k}_{\aleph_n}$ holds for all large n, and every stationary subset of $\aleph_n \cap \operatorname{cof}(\aleph_k)$ consists of approachable points. Then mutual stationarity holds for $\langle \aleph_n \cap \operatorname{cof}(\aleph_k) \mid k < n < \omega \rangle$.*

We build the ideals witnessing $\dagger$ via large cardinal embeddings as follows.

Lemma 2.9. *[2, Fact 2.14] Let $j : V \to M$ be an elementary embedding with $\operatorname{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$. Let $\mathbb{P} \in V$ be a poset and let G be generic for $\mathbb{P}$. Suppose that $j(\mathbb{P})$ projects to $\mathbb{P}$, so that every $j(\mathbb{P})/G$ generic contains j"G. Working in $V[G]$, for every $\gamma \in j(\kappa) \setminus \kappa$ and $r \in j(\mathbb{P})/G$, define an ideal $I_{\gamma,r}$ by*

$$I_{\gamma,r} = \{\dot{X}_G \mid r \Vdash_{j(\mathbb{P})/G} \gamma \notin j(\dot{X})\}.$$

Then this ideal is well defined and has the following properties:

- *$I_{\gamma,r}$ is κ-complete and nonprincipal.*
- *$I_{\gamma,r}$ is nonstationary iff $r \Vdash \gamma \in j(\dot{C})$ for every $\mathbb{P}$-name $\dot{C}$ for a club subset of κ.*
- *If $j(\mathbb{P})/\mathbb{P}$ is $(\mu+1)$-closed for some $\mu < \kappa$, then $I_{\gamma,r}$ is a $(\mu+1)$-closed ideal.*

To construct these projections, we will make heavy use of the following standard lemma regarding absorption of collapsing posets; it is an immediate corollary of [11, Lemma 3].

Lemma 2.10. *Let κ be regular. Let $\mathbb{P}$ be a κ-closed separative forcing notion, and let $\mathbb{Q} = Col(\kappa, \lambda)$ where λ is the cardinality of the set of dense subsets of $\mathbb{P}$. Then there is a forcing projection from $\mathbb{Q}$ to $\mathbb{P}$ with a κ-closed quotient.*

To verify that these ideals are nonstationary and meet the requisite stationary set, we use the following lemma. This argument is implicit in [2] and stated explicitly in [1].

Lemma 2.11. *[1, Lemma 2.9] Let $\lambda \geq 2^\kappa$, and let $j : V \to M$ be a λ-supercompactness embedding with critical point κ. Suppose $\mathbb{P}$ is a λ-cc poset such that $\mathbb{P}$ and j meet the hypotheses of Lemma 2.9, and $j(\mathbb{P})/\mathbb{P}$ is $(\mu+1)$-closed. Let G be generic for $\mathbb{P}$ over V. Let $S \subset \kappa$ be a stationary set in $V[G]$. Then there is a condition r and ordinal γ such that the ideal $I_{\gamma,r}$ given by Lemma 2.9 is $(\mu+1)$-closed and nonstationary, and $S \in I^+_{\gamma,r}$.*

Certain kinds of forcings will preserve the existence of these ideals.

Lemma 2.12. *Let V and W be two models of set theory, and let $\nu < \kappa$ be uncountable cardinals in both V and W. Suppose that $\mathcal{P}(\kappa)^V = \mathcal{P}(\kappa)^W$ and $\mathcal{P}(\mathcal{P}(\kappa))^V = \mathcal{P}(\mathcal{P}(\kappa))^W$. Then $V \models \dagger^\nu_\kappa$ if and only if $W \models \dagger^\nu_\kappa$.*

Proof. Suppose $\dagger^\nu_\kappa$ holds in one of the models; without loss of generality, assume $V \models \dagger^\nu_\kappa$. Then for all stationary $S \subseteq \kappa$, there is a nonstationary κ-complete $(\nu+1)$-closed ideal I_S in V with $S \in I^+_S$. Since W agrees with V on the powerset of κ, it has the same stationary subsets of κ, and for each stationary set S, W will also contain I_S. I_S will still be a nonstationary ideal in W, since W agrees with V on $\mathcal{P}(\kappa)$. Moreover, since W agrees with V on $\mathcal{P}(\mathcal{P}(\kappa))$, each I_S will be κ-complete and $(\nu+1)$-closed. □

As an immediate corollary, noting that we can code elements of $\mathcal{P}(\mathcal{P}(\kappa))$ as subsets of 2^κ, we obtain:

Lemma 2.13. *Suppose $\dagger^\nu_\kappa$ holds in V. Let $\mathbb{P}$ be a forcing that adds no new subsets of 2^κ, and let G be generic for $\mathbb{P}$ over V. Then in $V[G]$, $\dagger^\nu_\kappa$ still holds.*

Lemma 2.14. *Suppose $\dagger^\nu_\kappa$ holds in V. Let $\mathbb{P}$ be a forcing with size $< \kappa$, and let G be generic for $\mathbb{P}$ over V. Then in $V[G]$, $\dagger^\nu_\kappa$ still holds.*

Proof. Suppose $S \subseteq \kappa$ is a stationary set in $V[G]$. Working in V, let $\dot{S}$ be a $\mathbb{P}$-name for S. Since $|\mathbb{P}| < \kappa$, there is a generic condition $p \in G$ such that $S_0 := \{\alpha < \kappa \mid p \Vdash_{\mathbb{P}} \alpha \in \dot{S}\}$ is stationary in V.

By assumption, $V \models \dagger^\nu_\kappa$; let I_0 be the corresponding ideal with $S_0 \in I^+_0$. In $V[G]$, we generate an ideal I from I_0, defined by $I := \{X \subseteq \kappa \mid \exists X_0 \in I_0, X \subseteq X_0\}$. I is a κ-complete ideal. Since $S_0 \in I^+_0$ and $S_0 \subseteq S$, we see that $S \in I^+$. Now let $A \subseteq \kappa$ be nonstationary in $V[G]$. Since $|\mathbb{P}| < \kappa$, there is a nonstationary $A_0 \in V_0$ with $A \subseteq A_0$; we conclude that $A \in I$, so I is a nonstationary ideal. Finally, recall that I^+_0 had a $(\nu+1)$-closed dense subset D; this induces a closed dense subset of I^+. Since I has all the desired properties, we conclude that $V[G] \models \dagger^\nu_\kappa$. □

3. Extender-Based Forcing with Interleaved Collapses

In this section we define the forcing we will use. It is a variation on standard extender-based forcing, adding interleaved collapses. This construction was developed by Gitik [8]; our exposition will follow the description given in [13, Section 4].

3.1. **Setup.** Suppose $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals with supremum κ. Let $\mu = \kappa^+$ and $\lambda = \kappa^{++}$; suppose $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$. For all $n < \omega$, let $\sigma_n = \kappa_{n-1}^+$ (where we define $\kappa_{-1} := \aleph_0$).

Suppose κ_n is $(\lambda+1)$-strong for each $n < \omega$. In particular, there is a $(\kappa_n, \lambda+1)$-extender E_n with an associated embedding $j_n : V \to M_n$, such that M_n is a transitive class with ${}^{\kappa_n} M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$, and $j_n(\kappa_n) > \lambda$.

For each $n < \omega$ and $\alpha < \lambda$, define $E_{n,\alpha} := \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}$. If $\alpha \geq \kappa_n$, $E_{n,\alpha}$ is a nonprincipal κ_n-complete ultrafilter over κ_n. Note that E_{n,κ_n} is also normal.

Definition 3.1. For each $n < \omega$, define an ordering $\leq_{E_n}$ on λ as follows. We say $\beta \leq_{E_n} \alpha$ iff $\beta \leq \alpha$ and there exists a function $f : \kappa_n \to \kappa_n$ such that $j_n(f)(\alpha) = \beta$.

This gives a partial order on λ. For all $\beta \leq_{E_n} \alpha$ we fix a witnessing map $\pi_{\alpha,\beta} : \kappa_n \to \kappa_n$; we set $\pi_{\alpha,\alpha}$ to be the identity. Since the restriction of $\leq_{E_n}$ to κ_n^2 is the identity, we will only concern ourselves with $\lambda \setminus \kappa_n$.

3.2. **The Forcing.** The forcing is built up from two modules, called $\mathbb{Q}_{n0}$ and $\mathbb{Q}_{n1}$, which will be defined first. For all $n < \omega$, fix a map $s_n : \kappa_n \to \kappa_n$ representing μ in the normal ultrapower given by E_{n,κ_n}. That is, $j_n(s_n)(\kappa_n) = \mu$.

First we define the modules used in standard extender-based forcing, without interleaving any collapses. We will call these modules Q^*_{n0} and Q^*_{n1}. We follow the description given in [7, Section 2].

Definition 3.2. Q^*_{n1} is the set of partial functions from λ to κ_n of size $\leq \kappa$. We define an ordering $\leq^*_1$ on Q^*_{n1} by reverse inclusion.

Definition 3.3. Q^*_{n0} is the set of triples (a, A, f) satisfying the following conditions:

(1) $f \in Q^*_{n1}$.
(2) $a \subseteq \lambda$, such that:
 - $|a| < \kappa_n$
 - $a \cap \mathrm{dom}(f) = \emptyset$
 - a contains a $\leq_{E_n}$-maximal element, which we will denote $\mathrm{mc}(a)$.

(3) $A \in E_{n,\mathrm{mc}(a)}$
(4) For all $\alpha, \beta, \gamma \in a$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$ for every $\rho \in \pi_{\mathrm{mc}(a),\alpha}[A]$.
(5) For all $\alpha > \beta$ in a and all $\nu \in A$, $\pi_{\mathrm{mc}(a),\alpha}(\nu) > \pi_{\mathrm{mc}(a),\beta}(\nu)$.

Definition 3.4. We define the order $\leq^*_0$ on Q^*_{n0} by $(a, A, f) \leq^*_0 (b, B, g)$ iff:

(1) $f \supseteq g$
(2) $a \supseteq b$
(3) $\pi_{\mathrm{mc}(a),\mathrm{mc}(b)}[A] \subseteq B$.

With these defined, we build the modules $\mathbb{Q}_{n0}$ and $\mathbb{Q}_{n1}$ needed for defining the full forcing.

Definition 3.5. $\mathbb{Q}_{n0} = (Q_{n0}, \leq_{n0})$ is the set of $p := (a^p, A^p, f^p, F^{0p}, F^{1p}, F^{2p})$ satisfying the following conditions:

(1) (a^p, A^p, f^p) is an element of the module Q^*_{n0}. We also require that $\kappa_n, \mu \in a^p$ and that $a^p \cap \mu$ contains a $\leq_{E_n}$-greatest element denoted by $\mathrm{mc}(a^p \cap \mu)$.
(2) For $i < 3$, $\mathrm{dom}(F^{ip}) = \pi_{\mathrm{mc}(a^p),\mathrm{mc}(a^p\cap\mu)}[A^p]$. For $\nu \in \mathrm{dom}(F^{ip})$, setting $\nu_0 := \pi_{\mathrm{mc}(a^p\cap\mu),\kappa_n}(\nu)$, we have:

(a) $F^{0p}(\nu) \in Col(\sigma_n, <\nu_0)$;
(b) $F^{1p}(\nu) \in Col(\nu_0, s_n(\nu_0))$;
(c) $F^{2p}(\nu) \in Col(s_n(\nu_0)^{++}, <\kappa_n)$.

The ordering $\leq_{n0}$ is defined by $q \leq_{n0} p$ iff $(a^q, A^q, f^q) \leq_{\mathbb{Q}^*_{n0}} (a^p, A^p, f^p)$ and for each $\nu \in \operatorname{dom}(F^{iq})$, $F^{iq}(\nu) \supseteq F^{ip}(\nu')$, where $\nu' := \pi_{\operatorname{mc}(a^q \cap \mu), \operatorname{mc}(a^p \cap \mu)}(\nu)$.

Definition 3.6. $\mathbb{Q}_{n1} := (Q_{n1}, \leq_{n1})$ is the set of $p = (f^p, \rho^p, h^{0p}, h^{1p}, h^{2p})$, satisfying the following:

(1) f^p is a function from some $x \in [\lambda]^{\leq\kappa}$ to κ_n;
(2) $\rho^p < \kappa_n$ is an inaccessible cardinal;
(3) $h^{0p} \in Col(\sigma_n, <\rho^p)$;
(4) $h^{1p} \in Col(\rho^p, s_n(\rho^p))$;
(5) $h^{2p} \in Col(s_n(\rho^p)^{++}, <\kappa_n)$.

We define the ordering $\leq_{n1}$ as follows: $q \leq_{n1} p$ iff $f^q \supseteq f^p$, $\rho^p = \rho^q$, and for $i < 3$, $h^{iq} \supseteq h^{ip}$.

We codify the interaction between these modules in the following way.

Definition 3.7. Define $\mathbb{Q}_n := (Q_{n0} \cup Q_{n1}, \leq_n)$. We define the ordering $\leq_n$ by $q \leq_n p$ iff

- either $p, q \in Q_{ni}$ for some $i \in \{0, 1\}$ and $q \leq_{ni} p$, or
- $q \in Q_{n1}, p \in Q_{n0}$, and for some $\nu \in A^p$, $q \leq_{nq} p \frown \langle \nu \rangle$, where

$$p \frown \langle \nu \rangle := (f^p \cup \{\langle \beta, \pi_{\operatorname{mc}(a^p),\beta}(\nu)\rangle \mid \beta \in a^p\}, \bar{\nu}_0, F^{0p}(\bar{\nu}), F^{1p}(\bar{\nu}), F^{2p}(\bar{\nu})),$$

with $\bar{\nu} := \pi_{\operatorname{mc}(a^p), \operatorname{mc}(a^p \cap \mu)}(\nu)$.

Finally, we define the full poset $\mathbb{P}$.

Definition 3.8. Extender-Based Forcing with Collapses (EBFC) is the poset $\mathbb{P} = (P, \leq)$ defined as follows:

(1) Conditions in P are sequences $p = \langle p_n \mid n < \omega \rangle \in \prod_{n<\omega} Q_n$.
(2) For all $p \in P$,
  - There is $n < \omega$ such that $p_n \in Q_{n0}$;
  - For every $n < \omega$, if $p_n \in Q_{n0}$, then for all $m \geq n$, $p_m \in Q_{m0}$ and $a^{p_n} \subseteq a^{p_m}$.
(3) For all $p, q \in P$, $p \leq q$ iff $p_n \leq q_n$ for all $n < \omega$.

Definition 3.9. We define the *length* of a condition p by $l(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}$.

Let G be generic for $\mathbb{P}$. Note that in $V[G]$, GCH holds below $\aleph_\omega$, and SCH fails at $\aleph_\omega$. G adds (among other objects) a generic sequence $\langle \rho_n \mid n < \omega \rangle$, given by $\rho_n = \rho^p_n$ for some $p \in G$ with $l(p) \geq n$.

Let $\mathbb{C}^0_n = Col(\sigma_n, <\rho_n)$, $\mathbb{C}^1_n = Col(\rho_n, s_n(\rho_n))$, and $\mathbb{C}^2_n = Col(s_n(\rho_n)^{++}, <\kappa_n)$. Let $\mathbb{C}_n = \mathbb{C}^0_n \times \mathbb{C}^1_n \times \mathbb{C}^2_n$. For $n < m < \omega$, we will use $\mathbb{C}_{[n,m]}$ to represent the product $\prod_{n \leq i \leq m} \mathbb{C}_i$. G adds generics for these posets, which we will denote by C^0_n, C^1_n, and C^2_n.

We will make heavy use of two important properties of $\mathbb{P}$. The first fact follows immediately from a general lemma about Prikry-type forcings, [13, Lemma 3.14].

Fact 3.10. *Let G be $\mathbb{P}$-generic. If $a \in V[G]$ is a bounded subset of κ, then a is in $V[C_{[0,m]}]$ for some $m < \omega$.*

For our construction, we can refine this result further.

Corollary 3.11. *Let G be $\mathbb{P}$-generic. If $a \in V[G]$ is a subset of σ_{n+1}, then a is in $V[C_{[0,n]}]$.*

Proof. By the previous fact, a is in $V[C_{[0,m]}]$. For some $m < \omega$. If $m > n$, then since $\mathbb{C}_{[n+1,m]}$ is κ_n-distributive, a must be in $V[C_{[0,n]}]$. $\square$

The second property we will use is the existence of a generic cofinal sequence reflecting the properties of the sequence $\langle \kappa_n \mid n < \omega \rangle$. In particular:

Lemma 3.12. *Let $\langle A_n \mid n < \omega \rangle$ be a sequence with $A_n \in E_{n,\kappa_n}$. Then for all large n, $\rho_n \in A_n$.*

Proof. We claim that the set $D = \{p \mid \forall l.n \ \pi_{\mathrm{mc}(a_n^p \cap \mu),\kappa_n}[A_n^p] \subseteq A_n\}$ is dense.

Let p be a condition with length less than n. Let $A_n' = \pi^{-1}_{\mathrm{mc}(a_n^p),\kappa_n}[A_n^p]$. Then for each $n > \mathrm{lh}(p)$, $A_n' \in E_{n,\mathrm{mc}(a_n^p)}$. Extend p to a condition p' so that for all $n > \mathrm{lh}(p)$, $A_n^{p'} = A_n^p \cap A_n'$. Then $\pi_{\mathrm{mc}(a_n^p \cap \mu),\kappa_n}[A_n^p] \subseteq \pi_{\mathrm{mc}(a_n^p \cap \mu),\kappa_n}[A_n'] = A_n$.

Since D is dense and G is generic, there is some $p \in D \cap G$, with length k. Then for all $p' \leq p$ with length greater than k, $\rho_n^{p'} \in \pi_{\mathrm{mc}(a_n^p \cap \mu),\kappa_n}[A_n^{p'}] \subseteq \pi_{\mathrm{mc}(a_n^p \cap \mu),\kappa_n}[A_n^{p'}] \subseteq A_n$. Note also that $\rho_n^{p'}$ is the same for all $p' \in G$, so we conclude that for all $n > k$, $\rho_n \in A_n$. $\square$

4. Mutual Stationarity in EBF with Interleaved Collapses

In this section we give a new proof that mutual stationarity for a fixed cofinality can hold at $\aleph_\omega$ along with the failure of SCH at $\aleph_\omega$, using extender-based forcing with interleaved collapses. The failure of SCH in this construction is a stronger example of incompactness in this model than in the model of [1], since GCH will hold below $\aleph_\omega$.

4.1. Setup.

In a model V_0 of GCH, let $\langle \kappa_n \mid n < \omega \rangle$ and $\langle \sigma_n \mid n < \omega \rangle$ be increasing sequences of supercompact cardinals such that $\sigma_n < \kappa_n < \sigma_{n+1}$ for all $n < \omega$. Let $\kappa = \sup_n \kappa_n$, and let $\mu < \lambda$ both be supercompact with $\kappa < \mu$.

Let $\mathbb{L}_n$ denote the σ_n-directed closed Laver preparation to make κ_n indestructible under κ_n-directed closed forcings that preserve GCH. Each $\mathbb{L}_n$ will itself preserve GCH, and has size κ_n. This poset is constructed using standard techniques; for details see e.g. [13, Lemma 8.2].

In V_0, let $\mathbb{H}$ be the full-support iteration $\langle \mathbb{H}_n, \mathbb{H}(n) \mid n < \omega \rangle$, where for each n, $\mathbb{H}(n)$ is a $\mathbb{H}_n$-name for $Col(\kappa_{n-1}, < \sigma_n) * \dot{\mathbb{L}}_n$. (For the case $n = 0$, we set $\kappa_{-1} = \aleph_0$.) Let H be generic for $\mathbb{H}$ over V_0. In $V_0[H]$, force further with $Col(\kappa^+, < \mu) * \dot{Col}(\mu, < \lambda)$. Call the final model V. Note that in V, $\kappa^{++} = \mu$ and $\mu^+ = \lambda$.

Fact 4.1. *In V, the following hold:*

1. *GCH*
2. *$\sigma_n = \kappa_{n-1}^+$*
3. *Each κ_n is indestructible under κ_n-directed closed forcings that preserve GCH.*

Note that in this model, the supercompactness of each κ_n remains indestructible under κ_n-directed closed forcings that preserve GCH.

For each $n < \omega$, let $j_n : V \to M_n$ be a λ^+-supercompactness embedding with critical point κ_n, and let E_n be the associated $(\kappa_n, \lambda + 1)$-extender. In particular, we define $E_{n,\alpha} := \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}$. Note that E_{n,κ_n} is a normal measure on κ_n.

Working in V, let $\mathbb{P}$ be the extender-based Prikry forcing with interleaved collapses defined in the previous section, with respect to these extenders. Let G be generic for $\mathbb{P}$.

Let $\nu < \kappa$ be such that some condition forces $\check{\nu} = \dot{\aleph}_k$. For convenience, assume that we are forcing below this condition.

By Corollary 3.11, for any $n < \omega$, any bounded subsets of σ_{n+1} are present in $V[C_{[0,n]}]$. If $\nu < \tau$ are uncountable cardinals and $2^\tau \leq \sigma_{n+1}$, then by Lemma 2.12 $\dagger^\nu_\tau$ holds in $V[G]$ if and only if it holds in $V[C_{[0,n]}]$.

In the final model $V[G]$, there are five types of cardinals below κ that are preserved: $\sigma_n, \rho_n, s_n(\rho_n)^+, s_n(\rho_n)^{++}$, and κ_n. We have to verify the appropriate version of $\dagger$ for each type. The powerset of each has size at most σ_{n+1}, so by the previous paragraph it suffices to show that the appropriate version of $\dagger$ holds in $V[C_{[0,n]}]$.

In each case, we will use Lemma 2.10 to verify the existence of the forcing projections required to apply Lemma 2.9. The arguments involved are standard; see e.g. [4]. We will provide the details in the most complicated case (Lemma 4.2 and omit them for the rest.

4.2. **Cardinal type 1: σ_n.**

Lemma 4.2. *In $V[C^0_n]$, $\dagger^\nu_{\sigma_n}$ holds for all large n.*

Proof. Fix n such that $\kappa_n > \nu$. Recall that $V[C^0_n]$ is the extension of V_0 by $\mathbb{H} * \dot{Col}(\kappa^+, <\mu) * \dot{Col}(\mu, <\lambda) * \mathbb{C}^0_n$. This is a λ^+-cc poset. Note that in $V_0[\mathbb{H}_n]$, σ_n remains supercompact.

Let $j : V_0[\mathbb{H}_n] \to M$ be a λ^+-supercompactness embedding with critical point σ_n. The remainder of the setup poset $\mathbb{H}$ factors as $\mathbb{H}/\mathbb{H}_n = Col(\kappa_{n-1}, <\sigma_n) * \dot{\mathbb{L}}_n * \mathbb{H}/\mathbb{H}_{n+1}$. Consider $j(Col(\kappa_{n-1}, <\sigma_n))$. This poset factors as $Col(\kappa_{n-1}, <\sigma_n) \times Col(\kappa_{n-1}, [\sigma_n, j(\sigma_n)))$.

The forcing $\mathbb{L}_n * (\mathbb{H}/\mathbb{H}_{n+1}) * \dot{Col}(\lambda, <\mu) * \dot{Col}(\mu, <\lambda) * \dot{\mathbb{C}}^0_n$, is σ_n closed (and thus κ_{n-1}-closed), and has size λ. In particular, it has at most $2^\lambda = \lambda^+$ dense subsets. Applying Lemma 2.10, we see that $Col(\kappa_{n-1}, \lambda^+)$ projects to $\mathbb{L}_n * (\mathbb{H}/\mathbb{H}_{n+1}) * \dot{Col}(\lambda, <\mu) * \dot{Col}(\mu, <\lambda) * \dot{\mathbb{C}}^0_n$, with a κ_{n-1}-closed quotient.

Since j is a λ^+-supercompactness embedding, we have that $j(\sigma_n) > \lambda^+$. It follows that $Col(\kappa_{n-1}, [\sigma_n, j(\sigma_n)))$ projects to $Col(\kappa_{n-1}, \lambda^+)$ with a κ_{n-1}-closed quotient. From here, we see that there is a projection from $j(Col(\kappa_{n-1}, <\sigma_n))$ to $(\mathbb{H}/\mathbb{H}_n) * \dot{Col}(\lambda, <\mu) * \dot{Col}(\mu, <\lambda) * \dot{\mathbb{C}}^0_n$ with a κ_{n-1}-closed quotient. We conclude that $j((\mathbb{H}/\mathbb{H}_n) * \dot{Col}(\lambda, <\mu) * \dot{Col}(\mu, <\lambda) * \dot{\mathbb{C}}^0_n)$ projects to $(\mathbb{H}/\mathbb{H}_n) * \dot{Col}(\kappa^+, <\mu) * \dot{Col}(\mu, <\lambda) * \dot{\mathbb{C}}^0_n$ via a κ_{n-1}-closed quotient.

Applying Lemma 2.11 we see that in $V[\mathbb{C}^0_n]$, $\dagger^\nu_{\sigma_n}$ holds. □

Lemma 4.3. *In $V[C_n]$, $\dagger^\nu_{\sigma_n}$ holds for all large n.*

Proof. The remaining generics to add are C^1_n and C^2_n. Each is generic for a ρ_n-closed forcing, so the conclusion follows from Lemma 2.13. □

Lemma 4.4. *In $V[C_{[0,n]}]$, $\dagger^\nu_{\sigma_n}$ holds for all large n.*

Proof. Immediate from Lemma 2.14, noting that the remaining poset, $\mathbb{C}_{[0,n-1]}$, has size less than σ_n. □

4.3. **Cardinal type 2:** ρ_n**.**

Lemma 4.5. *Suppose $n < \omega$ is such that $\sigma_n > \nu$. Then*

$$\{\rho < \kappa_n \mid V \models \dagger^{\nu}_{\rho, Col(\sigma_n, <\rho) \times Col(\rho, s_n(\rho))}\} \in E_{n,\kappa_n}.$$

Proof. Consider the poset $Col(\sigma_n, < \kappa_n) \times Col(\kappa_n, < j_n(s_n)(\kappa_n)) = Col(\sigma_n, < \kappa_n) \times Col(\kappa_n, < \kappa^+)$. This poset is μ-cc, and there is a projection from $j_n(Col(\sigma_n, < \kappa_n) \times Col(\kappa_n, < \kappa^+))$ to $(Col(\sigma_n, < \kappa_n) \times Col(\kappa_n, < \kappa^+))$ with a σ_n-closed quotient. By Lemma 2.11, we conclude that $V \models \dagger^{\nu}_{\kappa_n, Col(\sigma_n, <\kappa_n)}$.

Since M_n is sufficiently closed, $M_n \models \dagger^{\nu}_{\kappa_n, Col(\sigma_n, <\kappa_n) \times Col(\kappa_n, <\kappa^+)}$. Then for E_{n,κ_n}-many ρ, $\dagger^{\nu}_{\rho, Col(\sigma_n, <\rho) \times Col(\rho, <s_n(\rho))}$ holds in V. Let A_n be the set of all such ρ. □

Lemma 4.6. *In $V[C^0_n \times C^1_n]$, $\dagger^{\nu}_{\rho_n}$ holds for all large n.*

Proof. Immediate from the previous lemma and Lemma 3.12. □

Lemma 4.7. *In $V[C_n]$, $\dagger^{\nu}_{\rho_n}$ holds for all large n.*

Proof. Follows from Lemma 2.13, noting that C^2_n is generic for a $s_n(\rho_n)^{++}$-closed forcing, and $2^{\rho_n} < s_n(\rho_n)^{++}$. □

4.4. **Cardinal type 3:** $s_n(\rho_n)^+$**.**

Lemma 4.8. *Suppose $n < \omega$ is such that $\nu < \kappa_n$. Then*

$$\{\rho < \kappa_n \mid V \models \dagger^{\nu}_{s_n(\rho)^+}\} \in E_{n,\kappa_n}.$$

Proof. Recall that V is the extension of $V_0[H]$ by $Col(\kappa^+, < \mu) * \dot{Col}(\mu^+, < \lambda)$. Note also that in $V_0[H]$, μ is supercompact.

In $V_0[H]$, let i be a λ-supercompactness embedding with critical point μ. Note that $Col(\kappa^+, < \mu) * \dot{Col}(\mu^+, < \lambda)$ is λ-cc, and $i(Col(\kappa^+, < \mu) * \dot{Col}(\mu^+, < \lambda))$ projects to $Col(\kappa^+, < \mu) * \dot{Col}(\mu^+, < \lambda)$ via a κ^+-closed quotient. Applying Lemma 2.11 we see that in V, $\dagger^{\nu}_{\mu}$ holds.

Since M_n is closed under λ-sequences, $\dagger^{\nu}_{\mu}$ holds in M_n as well. By definition, $j_n(s_n)(\kappa_n)^+ = \kappa^{++} = \mu$. We conclude that the set $\{\rho < \kappa_n \mid \dagger^{\nu}_{s_n(\rho)^+}\} \in E_{n,\kappa_n}$. □

Lemma 4.9. *In V, $\dagger^{\nu}_{s_n(\rho_n)^+}$ holds.*

Proof. Immediate from the previous lemma and Lemma 3.12. □

Lemma 4.10. *In $V[\mathbb{C}_{[0,n]}]$, $\dagger^{\nu}_{s_n(\rho_n)^+}$ holds.*

Proof. This follows from Lemma 2.13 and Lemma 2.14, noting that $|\mathbb{C}_{[0,n-1]} \times \mathbb{C}^0_n \times \mathbb{C}^1_n| < s_n(\rho_n)^+$ and $\mathbb{C}^2_n$ is $s_n(\rho_n)^{++}$-closed. □

4.5. **Cardinal type 4:** $s_n(\rho_n)^{++}$**.**

Lemma 4.11. *Suppose $n < \omega$ is such that $\nu < \kappa_n$. Then*

$$\{\rho < \kappa_n \mid \dagger^{\nu}_{s_n(\rho)^{++}, Col(s_n(\rho)^{++}, <\kappa_n)}\} \in E_{n,\kappa_n}.$$

Proof. Let K be generic for $Col(\kappa^+, < \mu)$, and recall that V is an extension of $V_0[H][K]$ by $Col(\mu, < \lambda)$. In $V_0[H][K]$, λ is supercompact. In this model, let i be a $j_n(\kappa_n)$-supercompactness embedding with critical point λ. Consider the poset $Col(\mu, < \lambda) * \dot{Col}(\lambda, < j_n(\kappa_n))$. This is a $j_n(\kappa_n)$-cc poset, and $i(Col(\mu, <$

$\lambda) * \dot{Col}(\lambda, < j_n(\kappa_n)))/Col(\mu, < \lambda) * \dot{Col}(\lambda, < j_n(\kappa_n))$ is μ-closed, so by Lemma 2.11, we conclude that $V \models \dagger^{\nu}_{\lambda, Col(\lambda, < j_n(\kappa_n))}$.

Since M_n is sufficiently closed, $\dagger^{\nu}_{\lambda, Col(\lambda, < j_n(\kappa_n))}$ will also hold in M_n. By definition, $j_n(s_n)(\kappa_n)^{++} = \kappa^{+++} = \lambda$. So by elementarity, the set $\{\rho < \kappa_n \mid \dagger^{\nu}_{s_n(\rho)^{++}, Col(s_n(\rho)^{++}, < \kappa_n)}\} \in E_{n,\kappa_n}$. □

Lemma 4.12. *In $V[C^2_n]$, $\dagger^{\nu}_{s_n(\rho_n)^{++}}$ holds.*

Proof. Immediate from the previous lemma and Lemma 3.12. □

Lemma 4.13. *In $V[\mathbb{C}_{[0,n]}]$, $\dagger^{\nu}_{s_n(\rho_n)^{+}}$ holds.*

Proof. This follows from Lemma 2.13 and Lemma 2.14, noting that $|\mathbb{C}_{[0,n-1]} \times \mathbb{C}^0_n \times \mathbb{C}^1_n| < s_n(\rho_n)^{++}$. □

4.6. **Cardinal type 5: κ_n.**

Lemma 4.14. *In $V[C^2_n]$, $\dagger^{\nu}_{\kappa_n}$ holds.*

Proof. In V, let $i : V \to M$ be a λ-supercompactness embedding with critical point κ. The lemma follows from Lemma 2.11, noting that $\mathbb{C}^2_n$ is κ_n-cc and $i(\mathbb{C}^2_n)$ can project to $\mathbb{C}^2_n$ via a $s_n(\rho_n)^{++}$-closed quotient. □

Lemma 4.15. *In $V[C_n]$, $\dagger^{\nu}_{\kappa_n}$ holds.*

Proof. The remaining piece of the forcing, $\mathbb{C}^0_n \times \mathbb{C}^1_n$, has size $< \kappa_n$, so we apply Lemma 2.14. □

Lemma 4.16. *In $V[C_{[0,n]}]$, $\dagger^{\nu}_{\kappa_n}$ holds.*

Proof. $V[C_{[0,n]}] = V[C_n][C_{[0,n-1]}]$. In $V[C_n]$, $\dagger^{\nu}_{\kappa_n}$ holds; since $|\mathbb{C}_{0,n-1}| < \kappa_n$, by Lemma 2.14 we have $\dagger^{\nu}_{\kappa_n}$ in the full extension. □

Putting all of this together, we can prove the full theorem:

Theorem 4.17. *Let G be generic for $\mathbb{P}$. In $V[G]$, mutual stationarity holds for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega \rangle$ for all $k < \omega$.*

Proof. Clearly SCH holds. In $V[G]$, each $\aleph_n$ is of the form $\sigma_i, \rho_i, s_i(\rho_i)^+, s_i(\rho_i)^{++}$, or κ_i for some $i < \omega$. Let $k < \omega$. By the previous lemmas, $\dagger^{\aleph_k}_{\aleph_n}$ holds in $V[C_{[0,n]}]$ for all $n > k$. By Lemma 2.12, it follows that $\dagger^{\aleph_k}_{\aleph_n}$ holds in V for all $n > k$. Since $\aleph_n \cap \mathrm{cof}(\aleph_k)$ is approachable for all $n > k + 1$ (see [14]), the theorem follows immediately from Corollary 2.8. □

5. Mutual Stationarity, Stationary Reflection, and the Failure of SCH

Before we proceed to the main theorem, it is worth reflecting on the arguments in the previous section. In particular, note that the proof of Theorem 4.16 only used a few key properties of the forcing. In particular, we needed the following:

1. Every bounded subset of κ is added by the collapses.
2. Every cardinal in the final model is either supercompact in an inner model, or reflects the properties of a supercompact in an inner model.
3. A suitable collapsing structure, to ensure that $\dagger^{\aleph_k}_{\aleph_n}$ will hold in $V[\mathbb{C}_{[0,n]}]$.

While the final property requires direct analysis of the collapsing poset to verify, the other two can be easily arranged for many Prikry-type posets (see e.g. standard Prikry forcing with interleaved collapses in [1]). This suggests that the same argument will work for similar constructions that share these properties, especially if they have the same collapsing structure.

In [13], an iteration scheme was developed for Prikry-type forcings with additional forcings interleaved. This scheme was used to force stationary reflection and the failure of SCH at $\aleph_\omega$. In this section, we show that the forcing used to obtain this result can be easily modified to obtain the same key properties we used in the previous section. As a consequence, we can use this modified version to force stationary reflection and the failure of SCH at $\aleph_\omega$, along with mutual stationarity at $\langle \aleph_n \cap \operatorname{cof}(\aleph_k) \mid k < n < \omega \rangle$ for all $k < \omega$. We restate the main theorem:

Theorem 5.1. *Suppose there is a sequence of indestructibly supercompact cardinals of length $\omega + 2$. Then there is a forcing extension in which the following properties hold:*

(1) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$
(3) Stationary reflection holds at $\aleph_{w+1}$
(4) Mutual stationarity holds at $\langle \aleph_n \cap \operatorname{cof}(\aleph_k) \mid k < n < \omega \rangle$ *for all* $k < \omega$.

Proof. Let V be the model described in Section 4. In this model, we build the forcing defined in [13] to kill all nonreflecting stationary sets while preserving the failure of SCH. In particular, we apply the following result:

Lemma 5.2. *[13] Suppose GCH holds, and suppose* $\langle \kappa_n \mid n < \omega \rangle$ *and* $\langle \sigma_n \mid n < \omega \rangle$ *satisfy the conclusion of Fact 4.1. Then there is a generic extension in which stationary reflection holds at* $\aleph_{\omega+1}$*, SCH fails at* $\aleph_\omega$*, and GCH holds below* $\aleph_\omega$*.*

Let $\mathbb{Q}$ be the forcing giving rise to this generic extension. $\mathbb{Q}$ is defined using a complicated iteration scheme. Informally, first one forces with EBFC, and then one iterates with Prikry-type forcings to kill nonreflecting stationary sets. Fortunately for the page count of this paper, we do not need a detailed description of the poset; instead, we will only require a few properties. See [13, Section 8] for details.

Fact 5.3. $\mathbb{Q}$ *projects to the poset* $\mathbb{P}$ *defined in Section 3. In fact, we can assume that the extenders used to build* $\mathbb{P}$ *in this projection are derived from* λ^+*-supercompactness embeddings as described in Section 4.*

Combining this fact with our analysis of $\mathbb{P}$ from the previous section, we obtain two immediate corollaries:

Corollary 5.4. $\mathbb{Q}$ *adds a sequence* $\langle \rho_n \mid n < \omega \rangle$ *with* $\rho_n < \kappa_n$*, such that if* $\langle A_n \mid n < \omega \rangle$ *is a sequence of sets in* E_{n,κ_n}*,* $\rho_n \in A_n$ *for all large* n*, where* E_{n,κ_n} *is the normal measure on* κ_n *as described in the previous section.*

Corollary 5.5. $\mathbb{Q}$ *adds a generic* $C_{[0,n]}$ *to the poset* $\mathbb{C}_{[0,n]}$ *described in the previous section.*

The analogous properties to Fact 3.10 and Corollary 3.11 also hold in this setting. As in the previous section, these follow from [13, Lemma 3.14].

Fact 5.6. *Let G be $\mathbb{Q}$-generic over V. If $a \in V[G]$ is a bounded subset of κ_n, then a is in $V[C_{[0,m]}]$ for some $m < \omega$.*

Fact 5.7. *Let G be $\mathbb{Q}$-generic over V. If $a \in V[G]$ is a subset of σ_{n+1}, then a is in $V[C_{[0,n]}]$.*

We force over V with $\mathbb{Q}$. The following lemma finishes the proof.

Lemma 5.8. *Let G be generic for $\mathbb{Q}$. Then in $V[G]$, the following properties hold:*

(1) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$
(3) Stationary reflection holds at $\aleph_{w+1}$
(4) Mutual stationarity holds at $\langle \aleph_n \cap \operatorname{cof}(\aleph_k) \mid k < n < \omega \rangle$ *for all* $k < \omega$.

Proof. The first three items follow immediately from Lemma 5.2, so it suffices to verify mutual stationarity. The argument is identical to the proof of Theorem 4.17, using Corollary 5.4, Fact 5.6 and Fact 5.7 in place of Lemma 3.12, Fact 3.10, and Corollary 3.11 respectively. □

□

As previously mentioned, the arguments in this paper are not dependent on the details of the forcing, and instead rely on a few core properties: the existence of a generic sequence reflecting the properties of supercompact cardinals, an analysis of bounded sets that allows us to only consider the interleaved collapses, and the structure of the collapses themselves. These properties are characteristic of Prikry-type forcings. While we modified the construction of [13] slightly, the basic forcing was unchanged; we simply introduced some extra large cardinals into the ground model and required slightly stronger properties from the extenders used. Thus we may hope that other more complicated Prikry-type forcings (in particular, iteration schemes like those described in [13]) will be equally susceptible to this kind of argument, allowing mutual stationarity to be neatly added to consistency results involving the failure of SCH without requiring significant modification of the forcing posets in question.

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