STAT 381-13605 Midterm Exam II 11/08/2006

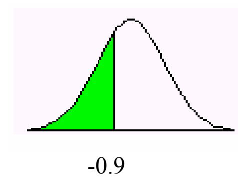
(40 points)

Solution

1. Find the following quantities:

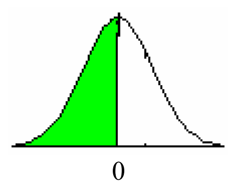
(a)[2pt] If Z ~ N (0, 1), calculate P (Z > - 0.9)

$P(Z > -0.9) = P(Z < 0.9) = \Phi(0.9) = 0.8159$



(b) [2pt] If X ~ N (2, 1), calculate P (X < 2)

$$P(X < 2) = P\left(Z < \frac{2-2}{1}\right) = P(Z < 0)$$
$$= \Phi(0) = 0.5$$

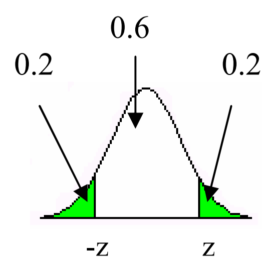


(c) [2pt] Find z such that P (|Z| > z) = 0.4, if Z ~ N (0, 1).

$P(|Z| > z) = 0.4$, i.e. $P(|Z| < z) = 0.6$

$\Phi(z) = P(Z < z) = 0.6+0.2=0.8$

$z= 0.84$



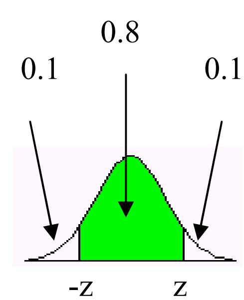
(d) [3pt] Find c such that P (|X-2| < c) = 0.8 for X ~ N (2, 4)

$P(|X-2| < c) = P\left(\left|\frac{X-2}{2}\right| < \frac{c}{2}\right) = 0.8$

Let $z = c/2$

$\Phi(z) = 0.8 + 0.1 = 0.9 \rightarrow z = 1.28$

$C=2 * 1.28 =2.56$



2. If X ~ U [a, b], then it has probability density function f(x)=1/(b-a), if a ≤ x < b; f(x)=0, otherwise.

(a) [4pt] Find the cumulative distribution function of X, F(x).

If $x<a$, then $F(x)=0$; if $x \geq b$, then $F(x)=1$.

If $a \leqslant x < b$, then

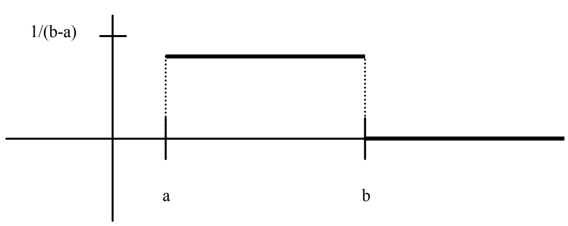
$$F(x)=\int_{-\infty}^{x} f(w)dw=\int_{-\infty}^{a} f(w)dw+\int_{a}^{x} f(w)dw=0+\int_{a}^{x}\frac{1}{b-a}dw=\left[\frac{w}{b-a}\right]_{a}^{x}=\frac{x-a}{b-a}$$

Hence the cumulative function of X is

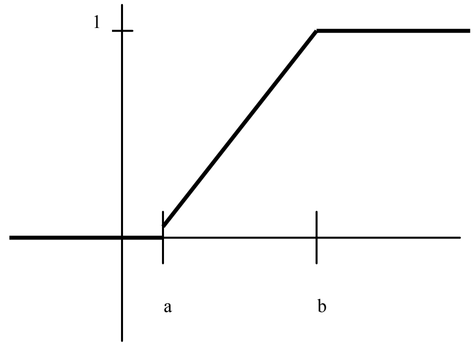
$$F(x)=\begin{cases} 0, & x\le a \\ \dfrac{x-a}{b-a}, & a<x\le b \\ 1, & x>b \end{cases}$$

(b) [2pt] Draw the graphs of p.d.f. and c.d.f. of X respectively.

P.D.F.



C. D. F.



3. The weight of a batch of parts is normally distributed with mean 100 gs and standard deviation 16 gs.

(a) [2pt] What is the mean and standard deviation of the average weight of a random sample of size 16?

Let X denote the population weight of the parts, then X ~ N (μ=100, $\sigma^2=16^2$)

The average weight is denoted by $\overline{X}$ of sample size n =16.

Mean E($\overline{X}$) = $\mu_{\overline{X}}$ = μ=100, Standard deviation: SD($\overline{X}$)= $\sigma_{\overline{X}}=\frac{\sigma}{\sqrt{n}}=\frac{16}{\sqrt{16}}=4$

(b) [3pt] What is the probability that a random sample of size 4 will have a sample mean between 99 gs and 101 gs?

$$P(99<\overline{X}<101)=P\left(\frac{99-100}{4}<\frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}}<\frac{101-100}{4}\right)=P(-0.25<Z<0.25)$$

$$=2\Phi(0.25)-1=2\cdot(0.5987)-1=0.1974$$

4. A furniture factory found that the number of reclamations concerning wood delivered by a certain supplier was on average four per year.

(a) [3pt]What is the probability of having no reclamation in all of the next year?

Let X denote the random variable for the number of reclamations per year,
Then X follows Poisson distribution with mean λ =4.

$$P(X=0) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{4^0 e^{-4}}{0!} = e^{-4} = 0.018$$

Or use Poisson Table, P(X=0) = F(0) = 0.018

(b) [2pt] What is the expected number of reclamations in three years?

Let Y be the number of reclamations in three years, then Y =3X

Expected number E(Y) = E(3X) = 3 E(X) = 3*4 = 12

Let W be the time until the first reclamation.
(c) [3pt] Find the probability density function for random variable W.

Random variable W is distributed as Gamma (α=1, $\beta=\lambda^{-1}$=1/4=0.25)

Density function $f(w) = \frac{1}{\Gamma(0) * 0.25} w^{1-1} e^{-w/0.25} = 4 \cdot e^{-4w}, 0 < w < \infty$

(d) [4pt]What is the probability that the time is longer than half a year, i.e. P(W>0.5)?

$$P(W > 0.5) = \int_{0.5}^{\infty} 4e^{-4y} dy = \left[-e^{-4y}\right]_{0.5}^{\infty} = \left[-e^{-\infty}\right] - \left[-e^{-4*0.5}\right]$$
$$= 0 - \left[-e^{-2}\right] = e^{-2} = 0.1353$$

5. Suppose the probability of producing a defective item is 0.05.

a). [3pt]What is the probability that exactly two is defective out of 6 items that are selected independently?

$$P(X=2)=\binom{6}{2}(0.05)^2(0.95)^4=0.305$$

Or use Binomial table

$$P(X=2)=P(X\le 2)-P(X\le 1)=F(2)-F(1)=0.9978-0.9672=0.306$$

b). [3pt] How many items do we have to produce so that the probability of producing at least one defective item is greater than 90%?

$P(X \geq 1) \geq .90$, then $1 - P(X=0) \geq .90$

$P(X=0) = 0.95^n$, then $1-0.95^n \ge 0.90$ i.e. $0.95^n \le 0.10$

$n \ge \log(0.10)/\log(0.95)$, so $n \geqslant 44.89 \nearrow n=45$

Bonus Question ☺ [2pt]

Suppose X is uniformly distributed on [1, 2], its density function f(x) =1, for 1 ≤ x < 2; 0 otherwise. Random variable Y is the inverse of X, i.e. Y= 1/X. Find the mean and the variance of Y.

Solution 1: Use c.d.f F(y) and p.d.f. f(y) to compute mean and variance of Y.

$1 \le x < 2 \rightarrow 0.5 \le x^{-1} < 1$, i.e. Y has positive density in the range of [0.5 , 1).

For $y < 0.5$, $F(y)=0$; $y \ge 1$, $F(y)=1$.

For $0.5 \le y < 1$, $F(y)=P(Y\le y)=P\left(X^{-1}\le y\right)=P\left(X\ge y^{-1}\right)=\int_{y^{-1}}^{1}(1)dx=1-y^{-1}$

Hence the density $f(y)=F'(y)=\begin{cases} y^{-2}, & 0.5\le y<1 \\ 0, & otherwise \end{cases}$

Mean $E(Y)=\int_{0.5}^{1} y\cdot f(y)dy=\int_{0.5}^{1} y\cdot y^{-2}dy=\int_{0.5}^{1} y^{-1}dy=[\ln(y)]_{0.5}^{1}=\ln(1)-\ln(0.5)=\ln 2$

and $E\left(Y^2\right)=\int_{0.5}^{1} y^2\cdot f(y)dy=\int_{0.5}^{1} y^2\cdot y^{-2}dy=\int_{0.5}^{1}(1)dy=0.5$

Variance of Y: $Var(Y)=E(Y^2)-[E(Y)]^2=0.5-(\ln 2)^2=0.0195$

Solution 2: Use the expectation of the function of a random variable.

Mean $E(Y)=E\left(X^{-1}\right)=\int_1^2 x^{-1}f(x)dx=\int_1^2 x^{-1}\cdot 1dx=[\ln x]_1^2=\ln 2-\ln 1=\ln 2$

$E\left(Y^2\right)=E\left(X^{-2}\right)=\int_1^2 x^{-2}f(x)dx=\int_1^2 x^{-2}\cdot 1dx=\left[-x^{-1}\right]_1^2=\left[-2^{-1}\right]-\left[-1^{-1}\right]=0.5$

Variance of Y: $Var(Y)=E(Y^2)-[E(Y)]^2=0.5-(\ln 2)^2=0.0195$