STAT 401 : Introduction to Probability
Midterm-II Exam
10:00am-10:50am, Nov. 13, 2009.

Solution

Name: __________________ UIN: __________________

Course CRN : 13614 (undergrad) 23465 (graduate)

1. This is a closed book, closed-notes examination. You may have a calculator.
2. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect, so show your reasoning.
3. Please provide the answers in the space provided. If you do not have enough space, please use the back of a nearby page. In this case, write a note to tell where to find the additional work; otherwise you may not get credit for the work.

Question	Points	Score
Question 1	24	
Question 2	32	
Question 3	24	
Question 4	20	
Total	100	

1. [24 points] Define the joint pmf of (X, Y) by

$$p(0,1) = p(0,2) = \frac{1}{6}, p(0,3) = \frac{1}{3}, p(1,1) = p(1,3) = \frac{1}{9}, p(1,2) = \frac{1}{9},$$

and $p(x, y) = 0$ for other (x, y).

(a) Find the marginal pmf of X.

(b) Calculate the conditional pmf of Y given $X = 0$.

(c) Find the conditional mean of Y given $X = 0$, $E(Y|X = 0)$.

Joint pmf:

$p(x,y)$		Y: 1	2	3	$p_X(x)$
X	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}+\frac{1}{6}+\frac{1}{3}=\frac{2}{3}$
	1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$

(a) Marginal pmf: $P(X=0) = p_X(0) = \frac{2}{3}$

$P(X=1) = p_X(1) = \frac{1}{3}$

(b) $$P(Y=y \mid X=0) = \begin{cases} \dfrac{p(0,1)}{p_X(0)} = \dfrac{1/6}{2/3} = \dfrac{1}{4}, & y=1 \\ \dfrac{p(0,2)}{p_X(0)} = \dfrac{1}{4}, & y=2 \\ \dfrac{p(0,3)}{p_X(0)} = \dfrac{1}{2}, & y=3 \end{cases}$$

(c) $E(Y \mid X=0)$

$$= \sum_{y=1}^{3} y \cdot P(Y=y \mid X=0)$$

$$= 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2}$$

$$= 2\frac{1}{4} = \frac{9}{4}$$

2. [32 points] Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $P(X_1 + X_2 \leq 1)$.

(b) Find the marginal densities for X_1 and X_2 respectively, and their means $E(X_1)$ and $E(X_2)$.

(c) Calculate its covariance $Cov(X_1, X_2)$.

(d) (undergraduate only) Suppose $Y_1 = X_1$, $Y_2 = X_1 + 2X_2$, find the joint pdf of (Y_1, Y_2).

(e) (graduate only) Suppose $Y_1 = X_1/X_2$, find its pdf. (Hint: set up a transformation $Y_1 = X_1/X_2$, $Y_2 = X_2$, and calculate joint pdf first)

(a). $P(X_1+X_2 \leq 1) = \int_0^{1/2} \int_{x_2}^{1-x_2} 3x_1 \, dx_1 \, dx_2 = \frac{3}{8}$

(b) Marginal density of X_1 and X_2.

$f_1(x_1) = \int_0^{x_1} 3x_1 \, dx_2 = 3x_1^2, \quad \forall\ 0 < x_1 < 1$

$f_2(x_2) = \int_{x_2}^{1} 3x_1 \, dx_1 = \frac{3}{2}(1-x_2^2), \quad \forall\ 0 < x_2 < 1$

Means: $E(X_1) = \int_0^1 x_1 \cdot 3x_1^2 \, dx_1 = \frac{3}{4}$

$E(X_2) = \int_0^1 x_2 \cdot \frac{3}{2}(1-x_2^2) \, dx_2 = \frac{3}{8}$

(c) $Cov(X_1, X_2) = E(X_1 \cdot X_2) - E(X_1) \cdot E(X_2)$

where $E(X_1 \cdot X_2) = \int_0^1 \int_0^{x_1} (x_1 x_2) \, 3x_1 \, dx_2 \, dx_1 = \frac{3}{10}$

$\therefore Cov(X_1, X_2) = \frac{3}{10} - \left(\frac{3}{4}\right) \times \left(\frac{3}{8}\right) = \frac{3}{160}$

(d) $\begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + 2X_2 \end{cases} \iff \begin{cases} X_1 = Y_1 \\ X_2 = (Y_2 - Y_1)/2 \end{cases} \quad \therefore J = \begin{vmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

Joint pdf of (Y_1, Y_2)

$f_{Y_1, Y_2}(y_1, y_2) = 3y_1 \cdot \frac{1}{2} = \frac{3}{2} y_1$

on support $\{(y_1, y_2) : 0 < \frac{1}{2}(y_2 - y_1) < y_1 < 1\}$

or $\{(y_1, y_2) : 0 < y_1 < 1, \ y_1 < y_2 < 3y_1\}$

(e) $\begin{cases} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} X_1 = Y_1 \cdot Y_2 \\ X_2 = Y_2 \end{cases}$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

Joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = 3(y_1 \cdot y_2) \cdot |J| = 3y_1 y_2^2$$

on support $\{(y_1, y_2): 0 < y_2 < y_1 y_2 < 1\}$

or $\{(y_1, y_2): 0 < y_2 < 1, 1 < y_1 < \frac{1}{y_2}\}$ or $\{0 < y_2 < \frac{1}{y_1}, \ y_1 > 1\}$

Marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \int_0^{\frac{1}{y_1}} 3y_1 y_2^2 \, dy_2 = y_1 \cdot \left(y_2^3 \Big|_0^{\frac{1}{y_1}} \right) = \frac{1}{y_1^2} \quad \forall \ y_1 > \phi$$

$$\left[\int_\phi^\infty f_{Y_1}(y_1)\, dy_1 = \int_1^\infty \frac{1}{y_1^2}\, dy_1 = \left(\frac{-1}{y_1}\right)\Big|_1^\infty = \frac{-1}{\infty} - \left(\frac{-1}{1}\right) = 1 \right]$$

3. [24 points] Let X follow normal distribution, $X \sim N(4, 4)$. Normal Table is attached.

   (a) Compute $P(2 < X < 5)$.

   (b) Find the first quartile Q_1 such that $P(X \leq Q_1) = 0.25$.

   (c) If $Y \sim N(4, 4)$ is independent of X, find the distribution of $Y - X$ and then calculate $P(Y > X)$.

$X \sim N(4,4) \quad \mu = 4, \ \sigma^2 = 4, \ \sigma = 2$

(a) $P(2 < X < 5) = P\left(\frac{2-4}{2} < \frac{X-\mu}{\sigma} < \frac{5-4}{2}\right)$

$= P(-1 < Z < .5)$ where $Z \sim N(0,1)$

$= \Phi(0.5) + \Phi(1) - 1 = .6915 + .8413 - 1 = 0.5328$

(b) Find z_p first such that $P(Z < z_p) = 0.25$

$1 - \Phi(z_p) = 0.75$ or $\Phi(-z_p) = 0.75$

From Normal table, $-z_p = .67$, $z_p = -0.67$

Given $\mu = 4$, $\sigma = 2$, $Q_1 = \mu + \sigma \cdot z_p = 4 + 2 \times (-.67) = 2.66$

(c) Y is independent of X, and $Y \sim N(4,4)$,

$\therefore \ Y - X \sim N(\mu_Y - \mu_X, \ Var(Y) + Var(X))$, i.e. $Y - X \sim N(0, 8)$

$P(Y > X) = P(Y - X > 0) = P\left(\frac{(Y-X) - \mu_{Y-X}}{\sigma_{Y-X}} > \frac{0-0}{\sqrt{8}}\right)$

$= P(Z > 0) = 0.5$

4. [20 points] Let the random variable X has a Binomial distribution

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \cdots, n$$

where$p > 0$, and $p + q = 1$. It is found that $P(X \geq 1) = 0.822$.

(a) If total trial number is $n = 6$, find the success rate p.

(b) If the success rate is known, $p = 0.15$, what is the minimum n so it satisfies $P(X \geq 1) = 0.822$?

(c) (graduate only). Given the MGF X $M_X(t) = [pe^t + q]^n$ for Binomial random variable. $X \sim$ Binomial $(n, 0.5)$. If Y is independent of X, and $X + Y \sim$Binomial $(m, 0.5)$, where $m > n$. What is the distribution of Y?

(a) $P(X \geqslant 1) = 0.822$ $\quad$ $1 - P(X=0) = 0.822$

$1 - 0.822 = 0.178$

$\Rightarrow \binom{6}{0} p^0 \cdot q^6 = 0.178$

$q^6 = 0.178$ $\quad$ $q = \sqrt[6]{0.178} = 0.75$

or $6 \ln q = \ln 0.178$ $\quad$ $q = \exp\left(\frac{1}{6} \ln(0.178)\right) = 0.75$

success rate $p = 1 - q = 0.25$

(b) Given $p = 0.15$, then $q^n = 0.178$ $\quad$ $q = 0.85$

$$n = \frac{\ln(0.178)}{\ln(0.85)} \approx 10.6, \quad n \nearrow 11$$

(c) For independent r.v. X and Y, MGF of $X+Y$.

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX}) \cdot E(e^{tY})$$

i.e. $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

It is known that $M_{X+Y}(t) = [0.5e^t + 0.5]^m$

$M_X(t) = [0.5e^t + 0.5]^n$

$$\therefore M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)} = [0.5e^t + 0.5]^{m-n} \Rightarrow Y \sim \text{Binomial}(n-m, 0.5)$$

and $m > n$