STAT 401 – Final Exam Sample

1. Two fair coins are to be tossed and the outcome is the ordered pair (face on the first coin, face on the second coin). Let the face "tail" be denoted by T and let the face "head" be denoted by H.

   (a) Describe the sample space of this random experiment.

   (b) Let X be the number of heads in the toss of two coins. Find the mean and variance of X.

   (c) Let Y be number of tails in the toss of two coins. Find the correlation coefficient of X and Y. (Hint: $Y = 2 - X$.)

2. Suppose we want to investigate the percentage of abused children in a certain population. The events of interest are: "a child is abused" (A) and its complement "a child is not abused" ($N = A^c$). For the purpose of this example, we will assume that $P(A) = 0.01$. The classification as to whether a child is abused or not is based upon a doctor's examination. Because doctors are not perfect, they sometimes classify an abused child (A) as one that is not abused (N_D, where N_D means classified as not abused by a doctor). On the other hand, doctors sometimes classify a nonabused child (N) as abused (A_D). Suppose these error rates of misclassification are $P(N_D|A) = 0.04$ and $P(A_D|N) = 0.05$.

   (a) Find $P(A_D)$, which is the probability that a child taken at random is classified as abused by a doctor.

   (b) Find $P(A|A_D)$, which is the conditional probability that a child is abused when the doctor classified the child as abused.

3. Let X_1, X_2, X_3 be i.i.d. with common pdf $f(x) = e^{-x},\ x > 0$, zero elsewhere.

   (a) Find the cdf $F_1(x)$ of X_1 .

   (b) Find $P(X_1 < 2X_2)$.

   (c) Find $P(X_1 < X_2|X_1 < 2X_2)$.

   (d) Find the joint pdf of $Y_1 = X_1$, $Y_2 = X_1 + 2X_2$, and $Y_3 = X_1 + 2X_2 + 3X_3$.

4. A certain job is completed in three steps in series. The means and standard deviations for the steps are (in minutes):

Step	Mean	Standard Deviation
1	17	2
2	13	1
3	13	2

   Assuming independent steps and normal distributions, compute the probability that the job will take less than 40 minutes to complete.

5. Let $X_1, \ldots, X_n$ be a random sample from a uniform(0, θ) distribution. Let $Y_n = \max\{X_1, \ldots, X_n\}$.

   (a) Find the cdf of X_1 .

   (b) Find the cdf of Y_n .

   (c) Find $E(Y_n)$. Is Y_n an unbiased estimator of θ?

   (d) Prove that $Y_n \xrightarrow{P} \theta$ as n goes to ∞ .

6. Let Y denote the sum of a random sample of size 50 from a distribution having pmf $p(x) = \frac{1}{5}$, $x = 1, 2, 3, 4, 5$, zero elsewhere. Compute an approximate value of $P(120 \le Y \le 130)$. (Hint: Use Central Limit Theorem and Continuity Correction.)

7. Let (X_1, X_2, X_3) have a multivariate normal distribution with mean 0 and variance-covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   (a). Calculate $P\left((X_1 - X_2 + X_3)^2 < 1\right)$

   (b). Let $Y = X_2 + X_3$. Find covariance of X_1 and Y, $Cov\,(X_1, Y)$, and corelation coefficient.

8. Let $X_{(n)}$ denote the maximum of a random sample from a distribution on $(0, 1)$, with cdf function $F\,(x) = x$. Find the limiting distribution of $Z_n = n[1 - X_{(n)}]$.

Answers:

1. (a). sample space={TT, TH, HT, HH}, (b). $\mu = 1$, $\sigma^2 = 0.5$, (c). $\rho = -1$.. **2.** (a). 0.06, (b). 0.1624.
3. (a). $F_1\,(x) = 1 - e^{-x}, x > 0$. (b). 2/3. (c).3/4.
(d). Joint pdf $g\,(y_1, y_2, y_3) = \frac{1}{6}\exp\left(-\left(y_1/2 + y_2/6 + y_3/3\right)\right)$, for $0 < y_1 < y_2 < y_3$.
4. (a). $P\,(X_1 + X_2 + X_3 < 40) = 0.1587$.

5. (a). cdf of X_1 : $F\,(x) = \begin{cases} 0, & x \le 0 \\ x/\theta, & 0 < x \le \theta \\ 1, & x > \theta \end{cases}$ (b). cdf of Y_n : $F_n\,(y) = \begin{cases} 0, & y \le 0 \\ y^n/\theta^n, & 0 < y \le \theta \\ 1, & y > \theta \end{cases}$

(c). We can show that $E\,(Y_n) = \frac{n}{n+1}\theta \ne \theta$, Y_n is not unbiased.

(d). For any $\varepsilon > 0, P\,(|Y_n - \theta| \ge \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n \to 0$, as $n \to \infty$.

6. $P(120 \le Y \le 130) = P(119.5 < Y < 130.5) \approx 0.0245$.

7. (a). $X_1 - X_2 + X_3 \sim N\,(0, 4)$, and $P\left((X_1 - X_2 + X_3)^2 < 1\right) = P\left(-\frac{1}{2} < Z < \frac{1}{2}\right) = 0.383$.

(b). $\mathbf{\Sigma}_{(X_1,Y)} = A{\cdot}\mathbf{\Sigma}{\cdot}A' = \begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix}$, where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. $Cov\,(X_1, Y) = 1$, $\rho = \frac{Cov(X_1,Y)}{\sqrt{Var(X_1)\cdot Var(Y)}} = \frac{1}{3}$.

8. For $x > 0$, $Z_n = n[1 - X_{(n)}]$, its cdf

$$P\,(Z_n \le x) = P\left\{n[1 - X_{(n)}] \le x\right\} = P\left\{X_{(n)} \ge 1 - \frac{x}{n}\right\}$$
$$= 1 - P\left\{X_{(n)} < 1 - \frac{x}{n}\right\} = 1 - \left(1 - \frac{x}{n}\right)^n \to 1 - e^{-x}\ .$$