STAT 401 MidtermII Sample —- Fall 2009

1. Define the joint pmf of (X, Y) by

$$p(0,1) = p(0,2) = \frac{1}{9}, p(0,3) = \frac{2}{9}, p(1,1) = p(1,3) = \frac{1}{6}, p(1,2) = \frac{2}{9},$$

and $p(x, y) = 0$ for other (x, y).

   (a) Find the marginal pmf of X.

   (b) Calculate the conditional pmf of Y given $X = 0$.

   (c) Find the conditional mean of Y given $X = 0$, $E(Y|X = 0)$.

2. Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 3x_1, & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) $P(X_1 + X_2 \leq 1)$.

   (b) Find the marginal densities for X_1 and X_2 respectively.

   (c) If X_1 and X_2 are independent? Prove or disprove it.

   (d) Calculate the correlation coefficient $\rho(X_1, X_2)$, given that $\mu_1 = \frac{3}{4}, \mu_2 = \frac{3}{8}$, $\sigma_1^2 = \frac{3}{80}, \sigma_2^2 = \frac{9}{320}$.

   (e) Suppose $Y_1 = X_1$, $Y_2 = X_1 + X_2$. Find the joint pdf of (Y_1, Y_2).

3. Let the random variable X has a Poisson distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, \cdots$$

It is found that $P(X = 1) = P(X = 2)$.

   (a) Find the parameter λ and calculate $P(X = 0)$.

   (b) Find the mean and variance. [Hint: MGF $M_X(t) = e^{\lambda(e^t - 1)}$] .

4. Let X follow normal distribution, $X \sim N(6, 1)$

   (a) Compute $P(5 < X < 7)$.

   (b) Find the quantile x_p such that $P(X \leq x_p) = 0.975$.

   (c) If $Y \sim N(7, 1)$ is independent of X, find the distribution of $Y - X$ and then calculate $P(Y > X)$.

Solution:

1. (a). $p_1(0) = \frac{4}{9}, p_1(1) = \frac{5}{9}$.

(b). $P(Y = y | X = 0) = \begin{cases} \frac{1}{4}, & y = 1 \\ \frac{1}{4}, & y = 2 \\ \frac{1}{2}, & y = 3 \end{cases}$.

(c). $E(Y|X = 0) = \sum_{y=1}^{3} y \cdot P(Y = y|X = 0) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} = \frac{9}{4}$.

2. (a). $P(X_1 + X_2 \leq 1) = \int_0^{1/2} \int_{x_2}^{1-x_2} 3x_1 dx_1 dx_2 = \frac{3}{8}$

(b). Marginal density function: $f_1(x_1) = \int_0^{x_1} 3x_1 dx_2 = 3x_1^2, \forall 0 < x_1 < 1$;

$f_2(x_2) = \int_{x_2}^{1} 3x_1 dx_1 = \frac{3}{2}\left(1 - x_2^2\right), \forall 0 < x_2 < 1$.

(c). Not independent. Hint: Joint and marginal density, or joint and marginal support.

(d). $E(X_1 X_2) = \int_0^1 \int_0^{x_1} (x_1 x_2) 3x_1 dx_1 dx_2 = \frac{3}{10}$.

Correlation coefficient: $\rho(X_1, X_2) = \frac{E(X_1X_2) - \mu_1 \cdot \mu_2}{\sigma_1 \cdot \sigma_2} = \frac{\sqrt{3}}{3}$.

(e). First find the inverse transformation $\begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \end{cases} \Leftrightarrow \begin{cases} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \end{cases}$, then the Jacobian determinant $J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$.

The joint density of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1, y_2 - y_1) \cdot |J| = 3y_1,$$

on its support $\{0 < y_1 < 1, y_1 < y_2 < 2y_1\}$.

3. (a). $\frac{\lambda^1}{1!} e^{-\lambda} = \frac{\lambda^2}{2!} e^{-\lambda} \Rightarrow \lambda = 2, \ (\lambda > 0)$, so $P(X = 0) = \frac{2^0}{0!} e^{-2} = e^{-2}$.

(b). $E(X) = M_X'(0) = 2, E\left(X^2\right) = M_X'(0) = 2^2 + 2$.

Mean and variance of $X : \mu = EX = 2, \sigma^2 = E\left(X^2\right) - [EX]^2 = 2$

4. (a). $P(5 < X < 7) = 0.6826$

(b). $\Phi(1.96) = 0.975$. The quantile is $x_p = \mu + \sigma \cdot z_p = 6 + 1 \cdot 1.96 = 7.96$.

(c). Mean and variance of $Y - X$: $E(Y - X) = 1, \text{Var}(Y - X) = \text{Var}(Y) + \text{Var}(X) = 2$, it implies that $Y - X \sim N(1, 2)$. Hence standardized r.v. is $Z = \frac{(Y-X)-1}{\sqrt{2}} \sim N(0, 1)$.

$P(Y > X) = P(Y - X > 0) = P\left(Z > -\frac{1}{\sqrt{2}}\right) = 0.7611$.