STAT 411 : Statistical Theory
Final Exam – 31576, 31577
December 10, 2010, 10:30 -12:30 PM

Name: ______________________________

UIN: ______________________________

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1. This is a closed book, closed-notes examination. You may have a calculator.
2. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect, so show your reasoning.
3. Please provide the answers in the space provided. If you do not have enough space, please use the back of a nearby page. In this case, write a note to tell us where to find the additional work; otherwise you may not get credit for the work.
4. In most cases the later parts of a question do not require the answers to earlier parts. You should try all parts of a problem even if you get stuck on an early part. If necessary, state and assume a value from an earlier part.

Question	Points	Score
Question 1	28	
Question 2	20	
Question 3	18	
Question 4	34	
Total	100	

1. [28 points] Given that random sample $X_1, X_2, \ldots, X_n$ is drawn from a Poisson distribution $X \sim Poisson\,(\theta)$, where $\theta > 0$ with probability function

$$f\,(x, \theta) = \frac{\theta^x}{x!} e^{-\theta}, x = 0, 1, 2, ...$$

It is known that $E\,(X) = Var\,(X) = \theta$.

(a). [8 points] Find the maximum likelihood estimator (mle) $\hat{\theta}$ of parameter θ.

(b). [8 points] Calculate the Fisher information $I\,(\theta)$ and Rao-Cramer lower bound. Is the mle estimator $\hat{\theta}$ efficient for θ?

(c) . [6 points] What is the asymptotic distribution of $\sqrt{n}\left(\hat{\theta} - \theta_0\right)$ for a given $\theta_0 > 0$?

(d) . [6 points] If we are interested in a new parameter $\eta = \theta^2$, find its mle $\hat{\eta}$. Is it unbiased for θ^2?

2. [20 points] Suppose $X_1, \ldots, X_n$ are iid with the pdf

$$f(x; \theta) = e^{-(x-\theta)}\ , \quad \theta \le x < \infty,$$

where $-\infty < \theta < \infty$.

(a). [8 points] Show that $X_{(1)} = \min\{X_1, \ldots, X_n\}$ is a sufficient statistic for θ.

(b) . [8 points] Is $X_{(1)}$ also a minimal sufficient statistic for θ?

(c) . [4 points] The parameter θ in the distribution function is a location parameter or a scale parameter? Why? Find its standard distribution.

3. [18 points] Sample $X_1, ..., X_n$ is following a Bernoulli distribution $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$ with $0 < \theta < 1$.

(a) . [8 points] Find a complete and sufficient statistic for θ.

(b). [6 points] Based on (a), construct a MVUE for θ.

(c). [4 points] Find a MVUE for $\theta\,(1-\theta)$.

4. [34 points] Sample $X_1, ..., X_n$ is following a normal distribution $N(\theta, 1)$,

$$f(x, \theta) = \left(\sqrt{2\pi}\right)^{-1} \exp\left\{-(x-\theta)^2/2\right\}$$

where $-\infty < \theta < +\infty$.

(a). [8 points] Based on Neyman-Pearson Theorem, find the best critical region for $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$ given significance level $0 < \alpha < 1$.

(b). [6 points] Calculate the power of the best test in (a) given that $n = 16$ and significance level $\alpha = 0.05$. [Stat tables are attached.]

(c). [6 points] Find a uniformly most powerful critical region of size $\alpha = 0.05$ for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$

(d). [8 points] Construct the likelihood ratio test statistic Λ for testing $H_0 : \theta = 0$ vs. $H_1 : \theta \neq 0$. Find its distribution or distribution of an equivalent statistic of Λ under the null hypothesis.

(e). [6 points] Find c such that the null hypothesis $H_0 : \theta = 0$ is rejected when $\Lambda \leq c$ with significance level $\alpha = 0.05$. [Stat tables are attached.]