Solution − STAT 411 Final − Fall 2010

1. [28 points] Given that random sample $X_1, X_2, \ldots, X_n$ is drawn from a Poisson distribution $X \sim Poisson(\theta)$, where $\theta > 0$ with probability function

$$f(x,\theta) = \frac{\theta^x}{x!}e^{-\theta}, x = 0, 1, 2, \ldots$$

It is known that $E(X) = Var(X) = \theta$.

(a). [8 points] Find the maximum likelihood estimator (mle) $\hat{\theta}$ of parameter θ.

$$l(\theta) = \log L(\theta) = \log\theta \cdot \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log x_i! - n\theta$$

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

$$\frac{\partial^2 l(\theta)}{\partial\theta^2} = \frac{-1}{\theta^2} \cdot \sum_{i=1}^{n} x_i < 0 \qquad \therefore \hat{\theta} = \bar{X} \text{ is the mle of } \theta.$$

(b). [8 points] Calculate the Fisher information $I(\theta)$ and Rao-Cramer lower bound. Is the mle estimator $\hat{\theta}$ efficient for θ?

$$I(\theta) = E\left[\frac{\partial \log f(x,\theta)}{\partial\theta}\right]^2 = E\left[\frac{X}{\theta} - 1\right]^2 = \frac{E(X-\theta)^2}{\theta^2} = \frac{Var(X)}{\theta^2} = \frac{1}{\theta}$$

R-C Bound is $\dfrac{1}{nI(\theta)} = \dfrac{1}{n(\frac{1}{\theta})} = \dfrac{\theta}{n}$

Since $E\hat{\theta} = E(\bar{X}) = \theta$ unbiased, and

$$Var(\hat{\theta}) = Var(\bar{X}) = \frac{Var(X)}{n} = \frac{\theta}{n} = \frac{1}{nI(\theta)}$$

$\therefore$ $\hat{\theta}$ is an efficient estimator of θ.

(c) . [6 points] What is the asymptotic distribution of $\sqrt{n}\left(\hat{\theta}-\theta_0\right)$ for a given $\theta_0 > 0$?

Under regularity conditions,

$$\sqrt{n}\,(\hat{\theta}-\theta_0)$$

$$\xrightarrow{\sim} N\left(0,\ I^{-1}(\theta_0)\right) = N(0,\ \theta_0)$$

(d) . [6 points] If we are interested in a new parameter $\eta=\theta^2$, find its mle $\hat{\eta}$. Is it unbiased for θ^2?

Based on Functional Invariance

$$\hat{\eta}_{mle} = \left(\hat{\theta}_{mle}\right)^2 = (\bar{X})^2 .$$

$$E(\hat{\eta}_{mle}) = E(\bar{X})^2 = Var(\bar{X}) + \left[E\bar{X}\right]^2$$

$$= \frac{\theta}{n} + \theta^2 \neq \theta^2$$

$\therefore$ $\hat{\eta}_{mle}$ is not unbiased,

But as $n\to\infty$, $E(\hat{\eta}_{mle}) \to \theta^2$, it is asymptotic unbiased.

2. [20 points] Suppose $X_1, \ldots, X_n$ are iid with the pdf

$$f(x;\theta) = e^{-(x-\theta)}\ , \quad \theta \le x < \infty,$$

where $-\infty < \theta < \infty$.

(a). **[8 points]** Show that $X_{(1)} = \min\{X_1, \ldots, X_n\}$ is a sufficient statistic for θ.

$$f(x_1,\cdots,x_n) = \prod_{i=1}^{n} e^{-(x_i-\theta)} \cdot I_{(x_i \geq \theta)}$$

$$= e^{-\sum_{i=1}^{n}(x_i-\theta)} \cdot I\{x_1,\cdots,x_n \geq \theta\} = e^{-\sum_{i=1}^{n}(x_i-\theta)} \cdot I\{x_{(1)} \geq \theta\}$$

$$= e^{-n\theta} \cdot I\{x_{(1)} \geq \theta\} \cdot e^{-\sum_{i=1}^{n} x_i}$$

$$= K_1(x_{(1)};\theta) \cdot K_2(x_1,\cdots,x_n)$$

Based on Factorization Theorem, $X_{(1)}$ is a sufficient stat. for θ.

(b). [8 points] Is $X_{(1)}$ also a minimal sufficient statistic for θ?

$$L(\theta) = e^{-\sum_{i=1}^{n}(x_i-\theta)} \cdot I\{x_{(1)} \geq \theta\}$$

$$\frac{\partial L(\theta)}{\partial\theta} = \frac{\partial \log L(\theta)}{\partial\theta} = -\sum_{i=1}^{n}(x_i-\theta) < 0$$

i.e. $L(\theta)$ is strictly decreasing on $\{\theta \leq x_{(1)}\}$

Hence $\hat{\theta} = X_{(1)}$ is the mle of θ.

From (a), $X_{(1)}$ is also sufficient for θ

$\Rightarrow$ $X_{(1)}$ is minimal sufficient for θ

(c) . [4 points] The parameter θ in the distribution function is a location parameter or a scale parameter? Why? Find its standard distribution.

$f(x) = e^{-(x-\theta)}, \quad x \geq \theta, \quad \theta \in R$

θ is a location paration where the location family

$$X = W + \theta$$

and its standard distribution

$$W \sim f_W(x) = e^{-x}, \quad x \geq 0$$

3. [18 points] Sample $X_1, ..., X_n$ is following a Bernoulli distribution $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$ with $0 < \theta < 1$.

(a) . [8 points] Find a complete and sufficient statistic for θ.

$$f(x, \theta) = \exp\left\{x \cdot \log\frac{\theta}{1-\theta} + \log(1-\theta)\right\}$$

$\mathcal{S} = \{0, 1\}$ is independent of $\Omega = (0, 1)$

It is a regular exponential family member with

$$K(x) = x, \quad p(\theta) = \log\frac{\theta}{1-\theta}, \quad q(\theta) = \log(1-\theta)$$

Hence its complete and sufficient statistic $Y = \sum_{i=1}^{n} K(X_i) = \sum_{i=1}^{n} X_i$

(b). [6 points] Based on (a); construct a MVUE for θ.

$$EY = E\left(\sum_{i=1}^{n} X_i\right) = n \cdot E(X) = n\theta$$

i.e. $E\left(\frac{Y}{n}\right) = \theta$

$\frac{Y}{n}$ is a 1-1 function of Y, a complete and sufficient stat, and also unbiased for θ.

Based on Lehmann - Scheffé Thm, $\frac{Y}{n}$ is the unique MVUE of θ.

(c). [4 points] Find a MVUE for $\theta(1-\theta)$.

$\bar{X}$ is MVUE for θ, also can show it is the mle.

$\tilde{\delta} = \bar{X}(1-\bar{X})$ is a good estimator for $\theta(1-\theta)$

$$E\bar{X}(1-\bar{X}) = E\bar{X} - E(\bar{X}^2) = E\bar{X} - [\mathrm{Var}\bar{X} + (E\bar{X})^2]$$

$$= \theta - \left[\frac{\theta(1-\theta)}{n} + \theta^2\right] = \frac{n-1}{n}\theta(1-\theta)$$

$\therefore \hat{\delta} = \frac{n}{n-1}\bar{X}(1-\bar{X})$ is unbiased for $\theta(1-\theta)$.

Based on extended Lehmann - Scheffé Thm,

$\hat{\delta} = \frac{n}{n-1}\bar{X}(1-\bar{X})$ is the unique MVUE for $\theta(1-\theta)$.

4. [34 points] Sample $X_1, ..., X_n$ is following a normal distribution $N(\theta, 1)$,

$$f(x, \theta) = \left(\sqrt{2\pi}\right)^{-1} \exp\left\{-(x-\theta)^2/2\right\}$$

where $-\infty < \theta < +\infty$.

(a). [8 points] Based on Neyman-Pearson Theorem, find the best critical region for $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$ given significance level $0 < \alpha < 1$.

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{L(0)}{L(1)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{x_i^2}{2}\right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{(x_i-1)^2}{2}\right\}} = \exp\left\{\frac{n}{2} - \sum_{i=1}^{n} x_i\right\}$$

Based on Neyman-Pearson Theorem ; critical region

$$\left\{\frac{L(0)}{L(1)} \le k\right\} \Leftrightarrow \left\{\vec{X} \in \underset{\uparrow}{\mathcal{C}}\right\} \Leftrightarrow \left\{\sum_{i=1}^{n} x_i \geq -\left(\frac{n}{2} - \log k\right)\right\}$$

$$\Leftrightarrow \mathcal{C} = \left\{\sum_{i=1}^{n} x_i \geq c^*\right\} = \left\{\bar{X} \geq c\right\}$$

where c is determined by $P_{\theta=0}\left\{\bar{X} \geq c\right\} = \alpha$, $\bar{X} \overset{H_0}{\sim} N(0, \frac{1}{n})$

$c \cdot \sqrt{n} = z_\alpha$, i.e. $\mathcal{C} = \left\{\bar{X} \geq z_\alpha/\sqrt{n}\right\}$ or $\left\{\sum_{i=1}^{n} x_i \geq z_\alpha \cdot \sqrt{n}\right\}$

(b). [6 points] Calculate the power of the best test in (a) given that $n = 16$ and significance level $\alpha = 0.05$. [Stat tables are attached.]

When $n=16$, $\alpha = 0.05$, the critical region is $(z_{.05} = 1.645)$

$$\mathcal{C} = \left\{\bar{X} \geq z_\alpha/\sqrt{n}\right\} = \left\{\bar{X} \geq \frac{1.645}{\sqrt{16}}\right\} = \left\{\bar{X} \geq 0.411\right\}$$

Power of test in (a) is $\bar{X} \overset{H_1}{\sim} N(\theta=1, \frac{1}{n})$.

$$P_{\theta=1}\left\{\bar{X} \geq c\right\} = P\left\{\frac{\bar{X} - \theta}{1/\sqrt{n}} \geq \frac{c-\theta}{1/\sqrt{n}}\right\}$$

$$= P\left\{Z \geq 4 \times (0.411 - 1)\right\} = P\left\{Z \geq -2.356\right\}$$

$$\approx 0.9908$$

(c). [6 points] Find a uniformly most powerful critical region of size $\alpha = 0.05$ for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$

$H_0: \theta = 0$ vs $H_1: \theta = \theta_1$ $(\theta_1 > 0)$

$$\frac{L(0)}{L(\theta_1)} = \exp\left\{ \frac{n}{2}\theta_1^2 - \sum_{i=1}^{n} x_i \cdot \theta_1 \right\} \le k$$

$\forall\ \theta_1 > 0$ then $\left\{ \frac{L(0)}{L(\theta_1)} \le k \right\} \Leftrightarrow \left\{ \sum_{i=1}^{n} x_i \geq -\left(\frac{n}{2}\theta_1 - \frac{1}{\theta_1}\log k\right) \right\}$

$\Leftrightarrow \{ \overline{X} \geq c \}$

where $P_{\theta=0}\{ \overline{X} \geq c \} = \alpha$

$c = \frac{z_\alpha}{\sqrt{n}} = \frac{1.645}{\sqrt{16}} = 0.411$ is the same for all $\theta_1 > 0$.

Hence $\mathcal{C} = \{ \overline{X} \geq 0.411 \}$ is the UMPT for $H_0: \theta = 0$ vs $H_1: \theta > 0$.

(d). [8 points] Construct the likelihood ratio test statistic Λ for testing $H_0 : \theta = 0$ vs. $H_1 : \theta \neq 0$. Find its distribution or distribution of an equivalent statistic of Λ. under H_0.

$$\Lambda = \frac{\max_{\theta \in \Omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)} = \frac{L(\theta = 0)}{L(\hat{\theta}_{mle})} \underset{\hat{\theta}_{mle} = \overline{X}}{=} \frac{L(0)}{L(\overline{X})} = \exp\left\{ -\frac{n}{2}\overline{X}^2 \right\}$$

$$\left[\begin{array}{l} l(\theta) = -n \log \sqrt{2\pi} - \sum_{i=1}^{n} (x_i - \theta)^2 / 2 \\ \left. \begin{array}{l} \frac{\partial l(\theta)}{\partial \theta} = + \sum_{i=1}^{n} (x_i - \theta) = 0, \quad \hat{\theta} = \overline{X} \\ \frac{\partial^2 l(\theta)}{\partial \theta^2} = -n < 0 \end{array} \right\} \Rightarrow \hat{\theta}_{mle} = \overline{X} \end{array} \right]$$

Hence $-2\log \Lambda = n\overline{X}^2 = \left(\frac{\overline{X} - 0}{\frac{1}{\sqrt{n}}} \right)^2 \overset{H_0}{\sim} \chi^2(1)$

since under $H_0: \theta = 0$, $\overline{X} \sim N(0, \frac{1}{n})$.

(*e*) . [6 points] Find c such that the null hypothesis $H_0 : \theta = 0$ is rejected when $\Lambda \leq c$ with significance level $\alpha = 0.05$. [Stat tables are attached.]

$$\mathcal{C} = \{ \Lambda \leq c \} = \{ -2 \log \Lambda \geq -2 \log c \}$$

Under H_0 : $-2 \log \Lambda \sim \chi^2(1)$

$$P_{H_0} \{ \Lambda \leq c \} = \alpha = 0.05$$

i.e. $P_{H_0} \{ -2 \log \Lambda \geq -2 \log c \} = 0.05$

$$\therefore \quad -2 \log c = \chi^2_{0.05}(1) = 3.841$$

$$c = \exp\left\{ -\frac{3.841}{2} \right\} = 0.1465$$

$\therefore$ Critical region of likelihood ratio test is

$$\{ \Lambda \leq 0.1465 \}$$