STAT 411 : Statistical Theory

Midterm-I, Feb. 18, 2015

Graduate Section: 19360

Name: ______________________ UIN: ______________________

---

1. This is a closed book, closed-notes examination.
2. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect.
3. Please provide the answers in the space provided. If you do not have enough space, please use the back of a nearby page. In this case, write a note to tell where to find the additional work.

Question	Points	Score
Question 1	30	
Question 2	30	
Question 3	40	
Total		

Problem 1. [30 points]

Suppose X follow an uniform distribution with parameter $\theta > 0$, i.e. $X \sim Unif\,(0, \theta)$, its density function is

$$f\,(x;\theta) = \frac{1}{\theta}, 0 < x \leq \theta.$$

A random sample $X_1, ..., X_n$ is drawn from the distribution.

(1). [6 points]. Find its cumulative distribution function $F\,(x)$.

$\forall\, 0 < x \le \theta$, $F(x) = \int_0^x \frac{1}{\theta}\, dt = \frac{x}{\theta}$

$\therefore$ cdf $F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{\theta} & 0 < x \le \theta \\ 1 & x > \theta \end{cases}$

(2). [10 points]. Let $X_{(n)}$ be the largest order statistics of the sample, find its density distribution.

Distribution Function of $Y = X_{(n)}$, $\qquad S_Y = (0, \theta]$

$\forall\, 0 < y \le \theta$, $F_Y(y) = P(Y \le y) = P(X_{(n)} \le y)$

$= P(X_1, \cdots, X_n \le y) = \prod_{i=1}^{n} P(X_i \le y)$

$= [F(y)]^n = \left(\frac{y}{\theta}\right)^n$

Density function

$f(y) = F'(y) = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \left(\frac{1}{\theta}\right) = \frac{n y^{n-1}}{\theta^n}$, $0 < y \le \theta$.

(3). [8 points] Find constant c such that $cX_{(n)}$ is unbiased for parameter θ.

$$\theta = E\left[c X_{(n)}\right] = c \cdot E X_{(n)} = c \int_0^{\theta} y \cdot \frac{n\, y^{n-1}}{\theta^{n-1}}\, dy$$

$$= \frac{n \cdot C \cdot}{\theta^{n-1}} \int_0^{\theta} y^n dy = \frac{nc}{\theta^{n-1}} \cdot \frac{y^n}{n+1}\Big|_0^{\theta}$$

$$= \frac{n}{n+1}\, \theta \cdot c$$

$$\Rightarrow \quad C = \frac{n+1}{n}$$

(4). [6 points]. Find the limiting distribution of $Z_n = n\left(\theta - X_{(n)}\right)$.

$S_Y = (0, \theta]$, $\quad S_{Z_n} = [0, n\theta)$, $\quad Z_n = n(\theta - Y)$

CDF $\quad F_{Z_n}(z) = P\left[n(\theta - Y) \le z\right]$

$$= P\left[Y \ge \theta - \frac{z}{n}\right]$$

$$= 1 - P\left(Y < \theta - \frac{z}{n}\right)$$

$$= 1 - F\left(\theta - \frac{z}{n}\right) = 1 - \left(\frac{\theta - \frac{z}{n}}{\theta}\right)^n$$

$$= 1 - \left(1 - \frac{z/\theta}{n}\right)^n$$

$$\xrightarrow[\text{as } n \to \infty]{} 1 - e^{-z/\theta} = F_Z(z),$$

where $Z \sim$ exponential distribution (θ).

$S_{Z_n} \to S_Z = (0, +\infty)$

Problem 2. [30 points]

Suppose X follows an exponential distribution, i.e. $X \sim Exp\,(1/\theta)$

$$f(x) = \theta e^{-\theta x}, x > 0. \qquad \sim \Gamma\left(1, \tfrac{1}{\theta}\right)$$

(1). [8 points] Calculate its moment generating function $M_X(t)$.

$$M_X(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} \cdot \theta \cdot e^{-\theta x}\, dx$$

$$= \theta \int_0^{\infty} e^{(t-\theta)x}\, dx$$

$$= \frac{\theta}{-(t-\theta)} \int_0^{\infty} e^{-v}\, dv \qquad \text{let } v = -(t-\theta)x$$

$$= \frac{\theta}{\theta - t} = \left(1 - \frac{t}{\theta}\right)^{-1}$$

(2). [8 points] Calculate its mean $E(X)$ and variance $Var\,(X)$.
[Gamma function: $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$.]

① $\quad EX = \int_0^{\infty} x \theta \cdot e^{-\theta x}\, dx = \frac{1}{\theta} \int_0^{\infty} v e^{-v}\, dv = \frac{\Gamma(2)}{\theta} = \frac{1}{\theta}$

$EX^2 = \int_0^{\infty} x^2 \theta\, e^{-\theta x}\, dx = \frac{1}{\theta^2} \int_0^{\infty} v^2 e^{-v}\, dv = \frac{\Gamma(3)}{\theta^2} = \frac{2}{\theta^2}$

$Var\,X = EX^2 - (EX)^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$

② $\quad EX = M'(0) = \frac{\theta}{(\theta - t)^2}\Big|_{t=0} = \frac{1}{\theta^2}$

$EX^2 = M''(0) = \frac{2\theta}{(\theta - t)^3}\Big|_{t=0} = \frac{2}{\theta^2}$

$Var\,X = EX^2 - (EX)^2 = \frac{1}{\theta^2}$

(3). [6 points] A random sample $X_1, ..., X_n$ is drawn from the distribution above, what is the distribution of $Y_n = X_1 + ... + X_n$?

[The MGF of Gamma distribution $\Gamma(\alpha, \beta)$, $M(t) = (1 - \beta t)^{-\alpha}$.]

$$M_{Y_n}(t) = E\left(e^{tY_n}\right) = E\left[e^{t(X_1+\cdots+X_n)}\right]$$

independent $= \prod_{i=1}^{n} E\left(e^{tX_i}\right)$

$$= \left[1 - \frac{t}{\theta}\right]^{-n}$$

$\therefore Y_n \sim \Gamma\left(n, \frac{1}{\theta}\right)$

(4). [8 points] Show that Y_n/n converges in probability to $1/\theta$.

$\frac{Y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$, $Var(\bar{X}) = \frac{Var X}{n} = \frac{1}{n\theta^2}$

X : mean $\frac{1}{\theta}$, variance $\frac{1}{\theta^2}$

① $P\left(\left|\bar{X} - \frac{1}{\theta}\right| > \varepsilon\right) \le \frac{Var(\bar{X})}{\varepsilon^2} = \frac{1}{n\theta^2\varepsilon^2} \to 0$

i.e. $\bar{X} \xrightarrow{P} \frac{1}{\theta}$

② By WLLN: $\bar{X} = \frac{Y_n}{n} \xrightarrow{P} EX = \frac{1}{\theta}$.

3. [40 points]. Consider a random sample $(X_1, \cdots, X_n)$ drawn from a normal population with mean θ and variance σ^2 (known)

$$f(x) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}.$$

(1). [12 points] Find the mle $\hat{\theta}$ for θ.

$L(\theta) = \prod_{i=1}^{n} f(x_i) = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}$

$l(\theta) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$

$\frac{\partial l(\theta)}{\partial\theta} = 0 \quad \Rightarrow \quad \frac{+2}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta) = 0$, $\quad \frac{\partial^2 l(\theta)}{\partial\theta^2} = \frac{-n}{\sigma^2} < 0$

$\Rightarrow \hat{\theta}_{mle} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$E\bar{X} = \theta$ Unbiased

(2). [12 points] Check if the mle $\hat{\theta}$ is efficient for θ.

$\frac{\partial \log f(x,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\theta)^2}{2\sigma^2}\right] = \frac{2(x-\theta)}{2\sigma^2} = \frac{x-\theta}{\sigma^2}$

$\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[\frac{x-\theta}{\sigma^2}\right] = \frac{-1}{\sigma^2}$

Fisher Information $I(\theta) = -E\left[\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2}\right] = \frac{1}{\sigma^2}$

$Var(\hat{\theta}) = Var(\bar{X}) = \frac{\sigma^2}{n}$

$\frac{1}{nI(\theta)} = \frac{1}{n\frac{1}{\sigma^2}} = \frac{\sigma^2}{n}$

$Var(\hat{\theta}) = \frac{[k'(\theta)]^2}{nI(\theta)} = \frac{1}{nI(\theta)}$

$k(\theta) = E(\bar{X}) = \theta$ unbiased $\quad Var(\hat{\theta}) = \frac{1}{nI(\theta)}$

$\therefore \hat{\theta}$ is an efficient estimator for θ.

(4). [10 points] What is the sampling distribution of the estimator? Construct a $100\,(1-\alpha)\,\%$ confidence interval for the unkown mean μ.

$X_1 \cdots X_n \sim N(\theta, \sigma^2)$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\frac{1}{n}\sum_{i=1}^{n} E(X_i)\ \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)\right)$

i.e. $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$

$P\left(-z_{\alpha/2} < \frac{\bar{X}-\theta}{\sqrt{\sigma^2/n}} < z_{\alpha/2}\right) = 1-\alpha$

C.I. for θ : $\bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

(5). [6 points] For a new parameter $\eta = \mu^3$, find its mle and its asymptotic distribution.

Based on functional invariant property of mle,

$\hat{\eta}_{mle} = (\hat{\theta}_{mle})^3 = \bar{X}^3$,

Delta method $g'(\theta) = (\theta^3)' = 3\theta^2$

$\sqrt{n}(\hat{\eta}_{mle} - \eta) \xrightarrow{\sim} N\left(0, \frac{[g'(\theta)]^2}{I(\theta)}\right)$

$\xrightarrow{\sim} N(0, 9\sigma^2\theta^4)$

STAT 411 : Statistical Theory

Midterm-I, Feb. 18, 2015

Undergraduate Section: 18086

Name: ______________________ UIN: ______________________

1. This is a closed book, closed-notes examination.
2. In order to receive full credit for a problem, you should show all of your work and explain your reasoning. Good work can receive substantial partial credit even if the final answer is incorrect.
3. Please provide the answers in the space provided. If you do not have enough space, please use the back of a nearby page. In this case, write a note to tell where to find the additional work.

Question	Points	Score
Question 1	30	
Question 2	30	
Question 3	40	
Total		