# 11

# Order Statistics from Independent Exponential Random Variables and the Sum of the Top Order Statistics

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**Abstract:** Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics from $n$ independent nonidentically distributed exponential random variables. We investigate the dependence structure of these order statistics, and provide a distributional identity that facilitates their simulation and the study of their moment properties. Next, we consider the partial sum $T_i = \sum_{j=i+1}^{n} X_{(j)}$, $0 \leq i \leq n-1$. We obtain an explicit expression for the cdf of $T_i$, exploiting the memoryless property of the exponential distribution. We do this for the identically distributed case as well, and compare the properties of $T_i$ under the two settings.

**Keywords and phrases:** Markov property, equal in distribution, simulation, mixtures, selection differential

## 11.1 Introduction

Let $X_1, \ldots, X_n$ be independent nonidentically distributed (inid) random variables (rvs), where $X_j$ is Exp($\lambda_j$), $j = 1, \ldots, n$; that is, the pdf of $X_j$ is given by

$$f_j(x) = \lambda_j e^{-\lambda_j x}, \ x \geq 0,$$

and the $\lambda_j$ are possibly distinct. Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics from this sample. We investigate their dependence structure and provide a distributional identity that facilitates their simulation and investigation of distributional and moment properties. This is done in Section 11.2.

The work in Section 11.3 is motivated by a personal communication from

Dr. Yang-Seok Choi who was interested in the distribution of

$$T_i = \sum_{j=i+1}^{n} X_{(j)},\ 0 \le i \le n-1. \tag{11.1}$$

There we obtain an explicit expression for the cdf of $T_i$. We also consider the independent identically distributed (iid) case and relate $T_i$ to a rv known as *selection differential* in the genetics literature. We then compare the properties of $T_i$ under the iid and inid models.

## 11.2 Distributional Representations and Basic Applications

We begin with a discussion of the stochastic structure of and distributional representations for the vector of order statistics $(X_{(1)}, \ldots, X_{(n)})$. When the $\lambda_j$ are identical and equal to, say $\lambda$, it is known that (see, e.g., David and Nagaraja, 2003, p. 18)

$$(X_{(i)}, i = 1, \ldots, n) \stackrel{d}{=} \frac{1}{\lambda}\left(\sum_{j=1}^{i} \frac{Z_j}{n-j+1}, i = 1, \ldots, n\right), \tag{11.2}$$

where the $Z_j$ are iid standard exponential (i.e., Exp(1)) rvs. This is known as Rényi's representation [Rényi (1953)].

Let $\mathbf{X} = (X_{(1)}, \ldots, X_{(n)})'$ and $\mathbf{Z} = (Z_1, \ldots, Z_n)'$, and define a vector $\boldsymbol{\alpha}_i = (\alpha_1, \ldots, \alpha_i, 0, \ldots, 0)'$ where $\alpha_j = 1/\{\lambda(n-j+1)\}$, $1 \le i, j \le n$. Then, $X_{(i)} \stackrel{d}{=} \boldsymbol{\alpha}_i{}'\mathbf{Z}$ and (11.2) can be expressed as

$$\mathbf{X} \stackrel{d}{=} \mathrm{C}\mathbf{Z}, \tag{11.3}$$

where C is the $n \times n$ matrix of constants whose $i$th row is $\boldsymbol{\alpha}_i{}'$. This relation is helpful in simulating all or a subset of order statistics from a random sample of size $n$ from an Exp($\lambda$) parent.

When the $\lambda_j$ are not identical, representations for the exponential order statistics do exist. Nevzorov (1984) shows that [see also Nevzorova and Nevzorov (1999)] the joint distribution of order statistics can be expressed as a mixture distribution with $n!$ components where the various component vectors are chosen with probability $p_l$ of picking certain permutation of the $\lambda_j$ for ordering the observed rvs. To be precise, Nevzorov shows that the cdf of $X_{(i)}$, the $i$th component of $\mathbf{X}$, can be expressed as a mixture cdf given by

$$F_{(i)}(x) = \sum_{l=1}^{n!} p_l F_l(x), \tag{11.4}$$

where

$$p_l = \frac{\lambda_1 \cdots \lambda_n}{(\lambda_{d(1)} + \cdots + \lambda_{d(n)})(\lambda_{d(2)} + \cdots + \lambda_{d(n)}) \cdots \lambda_{d(n)}} \tag{11.5}$$

and $F_l$ is the cdf of the rv

$$\frac{Z_1}{(\lambda_{d(1)} + \cdots + \lambda_{d(n)})} + \cdots + \frac{Z_i}{(\lambda_{d(i)} + \cdots + \lambda_{d(n)})}, \ 1 \le i \le n,$$

and the mixture includes all $n!$ vectors corresponding to the $n!$ permutations $(d(1), d(2), \ldots, d(n))$ of integers $1, 2, \ldots, n$.

Tikhov (1991) gave another, simpler, form of the above representation by introducing *antiranks* $D(1), \ldots, D(n)$ defined by

$$\{D(i) = m\} = \{X_{(i)} = X_m\}, \ \ 1 \le i, \ m \le n. \tag{11.6}$$

With these random subscripts, one can write the distributional equality

$$X_{(i)} \stackrel{d}{=} \frac{Z_1}{(\lambda_{D(1)} + \cdots + \lambda_{D(n)})} + \cdots + \frac{Z_i}{(\lambda_{D(i)} + \cdots + \lambda_{D(n)})}, 1 \le i \le n, \tag{11.7}$$

where the $Z_j$ are iid standard exponentials and are independent of the antirank vector $(D(1), \ldots, D(n))$. The form in (11.3) also holds in this case, with a modification that lets the elements of C to be rvs. Let us define a random vector $\mathbf{a}_i = (A_1, \ldots, A_i, 0, \ldots, 0)', 1 \le i \le n$, where

$$A_j = (\lambda_{D(j)} + \cdots + \lambda_{D(n)})^{-1}, 1 \le j \le n. \tag{11.8}$$

Then the following distributional equality holds:

$$\mathbf{X} \stackrel{d}{=} \mathbf{AZ}, \tag{11.9}$$

where $\mathbf{A}$ is an $n \times n$ random matrix whose $i$th row is $\mathbf{a}_i'$. The elements of $\mathbf{A}$ are independent of the vector $\mathbf{Z}$ whose components themselves are iid standard exponential rvs. The elements of $\mathbf{A}$ are functions of $A_1, \ldots, A_n$ that are dependent and depend on the distribution of $(D(1), \ldots, D(n))$, given by the $p_l$ in (11.5).

### 11.2.1 Remarks

**1.** The joint distribution of $(D(1), \ldots, D(n))$, given in (11.5), can be used to simulate this vector. We now describe how it can be done easily and more efficiently in a sequential manner. We start with $D(1)$; it is a discrete rv with support $\Omega_0 = \{1, 2, \ldots, n\}$ and $P(D(1) = i) = \lambda_i / (\sum_{j \in \Omega_0} \lambda_j)$. Once $D(1)$ is selected from this distribution, $D(2)$ is chosen from $\Omega_1 = \{1, 2, \ldots, n\} - \{D(1)\}$ using the probability distribution given by $P(D(2) = i) = \lambda_i / (\sum_{j \in \Omega_1} \lambda_j)$. In general, for $1 \leq k \leq n-1$, after $D(1), \ldots, D(k)$ are chosen, $D(k+1)$ is chosen from

$$\Omega_k = \{1, 2, \ldots, n\} - \{D(1), D(2), \ldots, D(k)\}$$

using the probabilities

$$P(D(k+1) = i) = \lambda_i \Bigg/ \left( \sum_{j \in \Omega_k} \lambda_j \right), \quad i \in \Omega_k, 1 \leq k \leq n-1.$$

**2.** The representation in (11.9) can be used to simulate exponential order statistics or functions of these order statistics. If the quantity of interest is a function of the first $i$ order statistics, one need to simulate only $D(1), \ldots, D(i)$ and these choices will determine the sum $\sum_{k=i+1}^{n} \lambda_{D(k)}$ that is needed to evaluate the observed values of $A_j, j \leq i$. Also, we need to simulate only $Z_k, 1 \leq k \leq i$.

**3.** The representation for the cdf of $X_{(i)}$ given in (11.4) and the distributional identity for the rv $X_{(i)}$ given in (11.7) have different purposes and applications. The former can be used to determine probabilities associated with $X_{(i)}$ assuming that the explicit form for $F_l$ is available, whereas the latter gives a handy framework for simulation. There is a distinction between (11.4) and an equality in distribution ($\stackrel{d}{=}$) relation obtained by replacing the cdfs with the associated rvs in that equation. Tikhov's (1991, p. 630) interpretation of Nevzorov's result makes this improper leap.

## 11.2.2 Applications

### Moments

We can use the distributional equality in (11.7) to obtain expressions for the moments of order statistics. Because

$$X_{(i)} \stackrel{d}{=} \sum_{j=1}^{i} A_j Z_j,$$

$A_j$ and $Z_j$ are independent, and the $Z_j$ are iid standard exponential, it follows that

$$E(X_{(i)}) = \sum_{j=1}^{i} E(A_j)$$

and

$$Var(X_{(i)}) = E(X_{(i)}^2) - \{E(X_{(i)})\}^2 = \sum_{j=1}^{i} E(A_j^2) + Var\left(\sum_{j=1}^{i} A_j\right),$$

upon simplification. Further, for $1 \le i < k \le n$,

$$\begin{aligned} Cov(X_{(i)}, X_{(k)}) &= Var(X_{(i)}) + \sum_{j=1}^{i}\sum_{l=i+1}^{k} Cov(A_j, A_l) \\ &= \sum_{j=1}^{i} E(A_j^2) + Cov\left(\sum_{j=1}^{i} A_j, \sum_{l=i+1}^{k} A_l\right). \end{aligned} \quad (11.10)$$

In the iid case, the $A_j$'s are all constants and $A_j = 1/\{\lambda(n-j+1)\}$, and the classical results follow immediately.

## Spacings of order statistics

The relation in (11.7) can also be used to study the distributional representations for spacings. For example,

$$X_{(i)} - X_{(i-1)} \stackrel{d}{=} A_i Z_i, \quad 2 \le i \le n,$$

and hence for $2 \le i \le n-1$,

$$Cov(X_{(i)} - X_{(i-1)}, X_{(i+1)} - X_{(i)}) = Cov(A_i Z_i, A_{i+1} Z_{i+1}) = Cov(A_i, A_{i+1}).$$

In the iid case, it is wellknown that the spacings are independent and thus are uncorrelated. It appears that the covariance is zero if and only if the $\lambda_i$ are identical. Such a conjecture is also made in Khaledi and Kochar (2000) and a proof is given of the claim for $n = 3$. (They actually prove a stronger result.) The case where $n > 3$ appears to be open.

## Other linear functions

For a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)'$, one can simulate $\boldsymbol{\beta}'\mathbf{X}$ values as $\boldsymbol{\beta}'\mathbf{AZ}$ using (11.9). For example, the $T_i$ in (11.1) can be simulated as the sum

$$T_i = (n-i)\sum_{j=1}^{i} A_j Z_j + \sum_{j=i+1}^{n} (n-j+1) A_j Z_j. \quad (11.11)$$

In the iid case, $T_i$ is related to the *selection differential*, given by

$$D_k = \frac{1}{\sigma}\left(\frac{1}{k}\sum_{j=n-k+1}^{n} X_{(j)} - \mu\right), \tag{11.12}$$

where $\mu$ and $\sigma$ are the mean and standard deviation of the parent population. For the Exp($\lambda$) parent, both these moments are $1/\lambda$. The rv $D_k$ is used to measure the improvement due to selection where the top values in the sample are selected and for small $k$ $(= n - i)$, it provides a good test for checking for outliers at the upper end of the sample.

Another linear function is the *total time on test* given by

$$\sum_{j=1}^{i} X_{(j)} + (n-i)X_{(i)},$$

and serves as the best estimator of $1/\lambda$ based on type II right censored sample in the iid case.

## 11.3 Sum of the Top Order Statistics

The following classical result (see, e.g., David and Nagaraja, 2003, pp. 137–138) is helpful in our pursuit of the cdf of the sum $T_i$.

**Lemma 11.3.1.** *Suppose $Z_r$, $r = 1, \ldots, m$, are independent standard exponential random variables and $c_r$'s are distinct positive numbers. Then*

$$P\left(\sum_{r=1}^{m}\left(\frac{Z_r}{c_r}\right) > z\right) = \sum_{r=1}^{m} w_r e^{-c_r z}, \quad z > 0,$$

*where*

$$w_r = 1\Big/\prod_{s\neq r}\left(1 - \frac{c_r}{c_s}\right),$$

*and the probability is 1 if $z < 0$.*

Now recall the representation (11.11) for $T_i$ where the joint distribution of the $A_j$ is as described in Section 11.2 and the $Z_j$ are iid standard exponential rvs.

### 11.3.1 The IID case

When the $X_i$ are identically distributed each being standard exponential, say, $A_j$ would be a constant $1/c_j$ where $c_j \equiv n - j + 1$. In that case,

$$T_i \stackrel{d}{=} (n-i)\sum_{j=1}^{i}\left(\frac{1}{c_j}\right) Z_j + W_i, \quad 0 \le i < n-1, \tag{11.13}$$

where $W_i$ is the sum of $(n-i)$ standard exponential rvs, and is a gamma$(n-i, 1)$ rv with pdf

$$f_i(w) = \frac{1}{(n-i-1)!} e^{-w} w^{n-i-1}, w > 0.$$

Thus, $T_0$ is a gamma$(n, 1)$ rv. Also, because $T_{n-1} = X_{(n)}$,

$$P(T_{n-1} > t) = 1 - (1 - e^{-t})^n, \ t > 0.$$

For $0 < i < n-1$, one can use Lemma 11.3.1 and conditioning argument in the representation (11.13) to obtain an explicit expression for the survival function of $T_i$ as follows:

$$\begin{aligned}
P(T_i > t) &= P\left((n-i)\sum_{j=1}^{i} \frac{1}{c_j} Z_j + Y_i > t\right) \\
&= \int_{y=0}^{t} \left(\sum_{j=1}^{i} \frac{1}{c_j} Z_j > \frac{1}{c_{i+1}}(t-y)\right) f_i(y)dy + P(Y_i > t) \\
&= \sum_{j=1}^{i} w_j \exp\left\{-c_j t / c_{i+1}\right\} \frac{1}{(n-i-1)!} \int_0^t \exp(d_j y) y^{n-i-1} dy \\
&\quad + \sum_{k=0}^{n-i-1} e^{-t} \frac{t^k}{k!}.
\end{aligned} \tag{11.14}$$

Here, $c_j = n - j + 1$,

$$d_j = \frac{c_j}{c_{i+1}} - 1 = \frac{i+1-j}{n-i} > 0.$$

The $w_j$ are obtained using Lemma 11.3.1, and have alternating signs. They are given by

$$w_j = \prod_{\substack{k=1 \\ \neq j}}^{i} \frac{n-k+1}{j-k} = \frac{1}{n-j+1} \frac{n!}{(n-i)!} \frac{(-1)^{i-j}}{(j-1)!(i-j)!}.$$

The pdf of $T_i$ can be obtained by differentiating (11.14). Upon some simplification the pdf can be expressed as

$$f_{T_i}(t) = \sum_{j=1}^{i} w_j \frac{c_j}{c_{i+1}} \exp\left\{-\frac{c_j}{c_{i+1}}t\right\} \frac{1}{(n-i-1)!} \int_0^t \exp(d_j y) y^{n-i-1} dy,$$

or as

$$\begin{aligned} f_{T_i}(t) &= n\binom{n-1}{i-1} \sum_{j=1}^{i} \binom{i-1}{j-1} (-1)^{i-j} \exp\left\{-\frac{n-j+1}{n-i}t\right\} \\ &\times \frac{1}{(n-i)!} \int_0^t \exp\left(\frac{i+1-j}{n-i}y\right) y^{n-i-1} dy. \end{aligned}$$

Nagaraja (1981) has obtained a similar expression for the pdf of $T_i/(n-i)$ in his study of the selection differential $D_k$ in (11.12) arising from a random sample from an exponential distribution. From Nagaraja (1982), one can obtain the asymptotic distribution of $T_i - (n-i)\log(n)$ if $n$ approaches infinity such that $k = n - i$ is held fixed. Becasue the exponential distribution is in the domain of attraction of the Gumbel distribution, the cdf of $T_i - k\log(n)$ converges to the following cdf for $k \geq 2$:

$$\frac{k^{k-1}}{(k-2)!} \sum_{j=0}^{k-1} \frac{\exp\{-(jx/k)\}}{j!} \int_0^\infty \exp\left\{-\exp\left(y - \frac{x}{k}\right)\right\} \exp\{-y(k-j)\} y^{k-2} dy.$$

Andrews (1996) has studied the finite-sample moment and distributional properties of the selection differential $D_k$ for the exponential and uniform parents. From his work, one can obtain explicit expressions for the first four moments of $T_i = (n-i)(\mu + \sigma D_{n-i})$ in the iid case. He also discusses asymptotes for the moments of $D_k$ when $k \approx np$, $0 < p < 1$, and the rate of convergence of the finite-sample moments.

### 11.3.2 The non-IID case

Let us assume that the $\lambda_j$ are all distinct. As in the iid case, we dispose of the special situations first. When $i = 0$,

$$T_0 = \sum_{j=1}^{n} X_{(j)} \equiv \sum_{j=1}^{n} X_j \stackrel{d}{=} \sum_{j=1}^{n} Z_j/\lambda_j.$$

Hence, Lemma 11.3.1 can be used directly to obtain an explicit expression for $P(T_0 > t)$.

When $i = n-1$, $T_i = X_{(n)}$ and hence

$$P(T_{n-1} > t) = 1 - \prod_{j=1}^{n} \left(1 - e^{-\lambda_j t}\right). \tag{11.15}$$

As we see below, for $1 \le i < n-1$, the expression for $P(T_i > t)$ is more involved.

For a given $j, 1 \le j \le n$, let $S(j)$ be a set with $(i-1)$ elements taken from $\{1,2,\ldots,n\} - \{j\}$. There are $\binom{n-1}{i-1}$ different choices for $S(j)$. For each such choice, let $\overline{S}(j) = \{1,2,\ldots,n\} - \{j\} - S(j)$.

**Theorem 11.3.1.** *Let $T_i$ be given by (11.1) with $1 \le i < n-1$. Then, for $t > 0$, $P(T_i > t)$ can be expressed as*

$$\sum_{j=1}^{n} \lambda_j \sum_{S(j)} \sum_{k \in \overline{S}(j)} w_k(\overline{S}(j)) e^{-\lambda_k t}$$
$$\times \int_0^{t/(n-i)} \prod_{m \in S(j)} (1 - e^{-\lambda_m x}) \exp\left\{ - \left[ \lambda_j + \sum_{r \in \overline{S}(j)} \lambda_r - (n-i)\lambda_k \right] x \right\} dx$$
$$+ \sum_{j=1}^{n} \lambda_j \sum_{S(j)} \int_{t/(n-i)}^{\infty} \prod_{m \in S(j)} (1 - e^{-\lambda_m x}) \exp\left\{ -(\lambda_j + \sum_{r \in \overline{S}(j)} \lambda_r) x \right\} dx, \tag{11.16}$$

*where*

$$w_k(\overline{S}(j)) = \frac{1}{\prod_{l \ne k \in \overline{S}(j)} \left(1 - \frac{\lambda_k}{\lambda_l}\right)}.$$

Proof. The joint pdf of $X_{(1)}, \ldots, X_{(n)}$ is the sum of $n!$ terms where each term has the form

$$\prod_{k=1}^{n} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k}, \quad 0 < x_1 < \cdots < x_n,$$

where $(r(1), \ldots, r(n))$ is a permutation of $(1, \ldots, n)$. Then

$$P(T_i > t) = \sum_{n!} \int \cdots \int_{\substack{0 < x_1 < \cdots < x_n < \infty \\ x_{i+1} + \cdots + x_n > t}} \prod_{k=1}^{n} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k} dx_k. \tag{11.17}$$

We split and group the $n!$ terms using the following procedure:

(a) We fix $X_{(i)} = x$ and its parameter $\lambda_j$, $j = 1, \ldots, n$.

(b) Given $j$, we fix the parameters associated with $X_{(1)}, \ldots, X_{(i-1)}$. There are

$$(n-1)\cdots(n-i+1) = \binom{n-1}{i-1}(i-1)!$$

such distinct ways of choosing their parameters.

(c) The remaining parameters associated with $X_{(i+1)}, \ldots, X_{(n)}$ can be ordered in $(n-i)!$ ways.

Let $S^o(j)$ be a typical (ordered) set in (b) and $\overline{S}^o(j)$ be a typical ordered set in (c). The expression for $P(T_i > t)$ given in (11.17) above can be written as

$$\sum_{j=1}^{n} \sum_{S^o(j)} \sum_{\overline{S}^o(j)} \int_{x=0}^{\infty} \lambda_j e^{-\lambda_j x} \left\{ \int \cdots \int_{0<x_1<\cdots<x_{i-1}<x} \prod_{k=1}^{i-1} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k} dx_k \right\}$$
$$\cdot \left\{ \int \cdots \int_{\substack{0<x<x_{i+1}<\cdots<x_n<\infty \\ x_{i+1}+\cdots+x_n>t}} \prod_{k=i+1}^{n} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k} dx_k \right\} dx. \tag{11.18}$$

For every unordered set $S(j)$ that leads to $S^o(j)$,

$$\sum_{S^o(j);\ S(j)\ \text{fixed}} \left\{ \int \cdots \int_{0<x_1<\cdots<x_{i-1}<x} \prod_{k=1}^{i-1} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k} dx_k \right\}$$

can be seen as

$$P(\max_{k \in S(j)} X_k < x) = \prod_{k \in S(j)} (1 - e^{-\lambda_k x}), \quad x > 0. \tag{11.19}$$

Further, in (11.18), for every unordered set $\overline{S}(j)$ that leads to $\overline{S}^o(j)$,

$$\sum_{\overline{S}^o(j);\ \overline{S}(j)\ \text{fixed}} \left\{ \int \cdots \int_{\substack{x<x_{i+1}<\cdots<x_n<\infty \\ x_{i+1}+\cdots+x_n>t}} \prod_{k=i+1}^{n} \lambda_{r(k)} e^{-\lambda_{r(k)} x_k} dx_k \right\}$$

can be expressed as

$$\sum_{\overline{S}^o(j);\overline{S}(j)\text{fixed}} e^{-x \sum_{r \in \overline{S}(j)} \lambda_r}$$
$$\cdot \left\{ \int \cdots \int_{\substack{0<y_{i+1}<\cdots<y_n<\infty \\ y_{i+1}+\cdots+y_n>t-(n-i)x}} \prod_{k=i+1}^{n} \lambda_{r(k)} e^{-\lambda_{r(k)} y_k} dy_k \right\}, \tag{11.20}$$

by taking $y_k = x_k - x$, $k = i+1, \ldots, n$. The multiple integral in (11.20), when summed over $\overline{S}^o(j)$ for a fixed $\overline{S}(j)$, represents

$$P(Y_{(1)} + \cdots + Y_{(n-i)} > t - (n-i)x)$$

where $Y_{(1)}, \ldots, Y_{(n-i)}$ are the sample order statistics generated from $(n-i)$ independent exponential rvs having $\exp(\lambda_r)$distribution, $r \in \overline{S}(j)$. Thus, the above expression is nothing but

$$P\left(\textstyle\sum_{r \in \overline{S}(j)} Y_r > t - (n-i)x\right) = P\left(\textstyle\sum_{r \in \overline{S}(j)} \frac{1}{\lambda_r} Z_r > t - (n-i)x\right), \tag{11.21}$$

where the $Z_r$ are iid standard exponential rvs. Thus, in view of Lemma 11.3.1, for a fixed $x$ and $\overline{S}(j)$, the expression in (11.21) reduces to

$$\sum_{r\in\overline{S}(j)} w_r(\overline{S}(j))e^{-\lambda_r\{t-(n-i)x\}}$$

if $x < t/(n-i)$, where the $w_r(\overline{S}(j))$ are as given in the theorem. The expression in (11.21) is clearly 1 if $x \geq t/(n-i)$.

Combining the above with (11.19) and (11.20), and recalling (11.18), we are led to the expression for $P(T_i > t)$ given in (11.16). ■

## Notes

**1.** The first summation in (11.16) above has $n \times \binom{n-1}{i-1} \times (n-i)$ distinct terms and the second summation has $n \times \binom{n-1}{i-1}$ terms.

**2.** The form given by (11.16) holds when $i = n-1$ as well. In that case $\overline{S}(j)$ has only one element, $w_k(\overline{S}(j)) = 1$, and $\sum_{r\in\overline{S}(j)} \lambda_r - (n-i)\lambda_k = 0$ in the above expression. However, the expression given by (11.12) is much easier to work with.

**3.** If some of the $\lambda_r$'s coincide, one could use limiting argument to obtain the relevant expression for $P(T_i > t)$. The extreme setup of this type is the iid case.

**4.** The distribution of the random variable $T_i$ is helpful in finding probabilities of interest in the performance analysis of multiple antenna systems. See for example, Choi *et al.* (2003). There, the inid case is of interest.

### 11.3.3 The IID case vs. the INID case

It would be interesting to study the changes in the distributional properties of $T_i$ as one moves from the iid case to the inid case. Of course, the additional complications that arise in the expression for the cdf in the inid case are evident in the above discussion. The question of interest could be in terms of stochastic comparisons. For example, how do the cdf of $T_i$ in the inid case compare with the one in the iid case?

Proschan and Sethuraman (1976) obtained a majorization result for order statistics from heterogeneous populations with proportional hazard functions. They showed that if the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)'$ majorizes $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)'$, $X_i$ is $\exp(\lambda_i)$, $Y_i$ is $\exp(\nu_i)$, and they are all mutually independent, then $(X_{(1)}, \ldots, X_{(n)})$ is stochastically larger than $(Y_{(1)}, \ldots, Y_{(n)})$. Without loss of generality, we can take $\lambda_1 \geq \cdots \geq \lambda_n$ and $\nu_1 \geq \cdots \geq \nu_n$, then the first vector majorizes the second if $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \nu_j$ for $1 \leq i < n$, and equality holds when $i = n$. This means any monotonically increasing function of order statistics is stochastically larger with parameter vector $\boldsymbol{\lambda}$ than with $\boldsymbol{\nu}$, and in particular, this property holds for $T_i$. The iid case corresponds to the vector $(\lambda, \ldots, \lambda)'$ and is majorized by any $\boldsymbol{\lambda}$ with at least two distinct components. Thus, $T_i$ will have a larger mean under heterogeneity than under homogeneity when the sum of the hazard rates remains the same. But, then one has to keep in mind that

$$\begin{aligned} E(X_1 + \cdots + X_n) \equiv E(T_0) &= \frac{n}{\lambda} \text{ (iid case)} \\ &= \sum_{i=1}^{n} \frac{1}{\lambda_i} \text{ (inid case).} \end{aligned}$$

When $\sum \lambda_i = n\lambda$, from the "arithmetic mean-harmonic mean inequality," it is clear that the mean of the sample average ($= T_0/n$) in the iid case is itself (much) smaller than its mean in the inid case. Thus, a similar result for $T_i$ when $i > 0$ is hardly surprising given that components of $T_i$ tend to be those $X_j$ with larger means or smaller hazard rates.

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