

## 2<sup>k</sup> Factorial Models

Design of Experiments - Montgomery  
Chapter 6

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## 2<sup>k</sup> Factorial Design

- Each factor has two levels (often labeled + and -)
- Very useful design for preliminary analysis
- Can “weed out” unimportant factors
- Also allows initial study of interactions
- For general two-factor factorial model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

- Have  $1 + (a - 1) + (b - 1) + (a - 1)(b - 1)$  parameters
- 2<sup>2</sup> factorial has only four parameters

$$\begin{matrix} \alpha_1 = -\alpha_2 & & \beta_1 = -\beta_2 \\ (\alpha\beta)_{11} = (\alpha\beta)_{22} & (\alpha\beta)_{12} = -(\alpha\beta)_{21} & (\alpha\beta)_{21} = -(\alpha\beta)_{12} \end{matrix}$$

- Can study 2 factors through repeated 2<sup>2</sup> designs
  - Select different levels of each factor

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## Design for 2<sup>2</sup> Factorial

- Label the levels of factors *A* and *B* using + and -
- Factors need not be on numeric scale
- There are 4 experimental combinations labeled

| <i>A</i> | <i>B</i> | Symbol |
|----------|----------|--------|
| -        | -        | (1)    |
| +        | -        | a      |
| -        | +        | b      |
| +        | +        | ab     |

- Can express combination in terms of model params

$$\begin{aligned} ab = (+, +) : E(y) &= \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\ a = (+, -) : E(y) &= \mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21} \\ b = (-, +) : E(y) &= \mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12} \\ (1) = (-, -) : E(y) &= \mu + \alpha_1 + \beta_1 + (\alpha\beta)_{11} \end{aligned}$$

$$\begin{aligned} ab = (+, +) : E(y) &= \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\ a = (+, -) : E(y) &= \mu + \alpha_2 - \beta_2 - (\alpha\beta)_{22} \\ b = (-, +) : E(y) &= \mu - \alpha_2 + \beta_2 - (\alpha\beta)_{22} \\ (1) = (-, -) : E(y) &= \mu - \alpha_2 - \beta_2 + (\alpha\beta)_{22} \end{aligned}$$

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## Estimating Model Parameters

- Have four equations and four unknowns

$$\begin{aligned} \hat{\mu} &= \frac{ab + a + b + (1)}{4n} \\ \hat{\alpha}_2 &= .5 \left( \frac{ab+a}{2n} - \frac{b+(1)}{2n} \right) = \frac{ab + a - b - (1)}{4n} \\ \hat{\beta}_2 &= .5 \left( \frac{ab+b}{2n} - \frac{a+(1)}{2n} \right) = \frac{ab - a + b - (1)}{4n} \\ \widehat{(\alpha\beta)}_{22} &= .5 \left( \frac{ab-b}{2n} - \frac{a-(1)}{2n} \right) = \frac{ab - a - b + (1)}{4n} \end{aligned}$$

where *n* represents the number of replications of each combination and *ab*, *b*, etc represent the sample sums of each combination.

- Main effect of *A* defined as  $2\hat{\alpha}_2$
- Main effect of *B* defined as  $2\hat{\beta}_2$
- Interaction effect defined as  $2(\hat{\alpha\beta})_{22}$
- Each effect function of  $\pm 1$ (combination sums)

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## Using Effects To Draw Inference

- Look at magnitude and direction of effects
- ANOVA then used to verify conclusions (Chapt 5)
- Effects and contrasts related as shown

| Contrast coefficients |     |    |    |    |
|-----------------------|-----|----|----|----|
| Effect                | (1) | a  | b  | ab |
| A                     | -1  | +1 | -1 | +1 |
| B                     | -1  | -1 | +1 | +1 |
| AB                    | 1   | -1 | -1 | +1 |

- AB coefficients simply product of A and B coeffs
- Sum of squares related to model effects

$$SS_{\text{Contrast}} = \frac{\left(\sum c_i y_i\right)^2}{n \sum c_i^2} \quad i = 1, 2, 3, 4$$

$$SS_A = \frac{(ab+a-b-(1))^2}{4n} = 4n\hat{\alpha}_2^2$$

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## Regression / Response Surface

- Code each factor as  $\pm 1$  where +1 is high level
- $x_1$  associated with Factor 1 = A
- $x_2$  associated with Factor 2 = B
- Regression Model

$$y_{ijk} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon_{ijk}$$

- Can show

$$\beta_0 = \hat{\mu} \quad \beta_1 = \hat{\alpha}_2 \quad \beta_2 = \hat{\beta}_2$$

$$\beta_{12} = (\widehat{\alpha\beta})_{22}$$

- Parameters based on unit increase (not -1 to +1)
- Can use regression model to form contours
- No interaction  $\rightarrow$  linear relationship
- Can often see direction for improvement in response

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## Hormone Example

| Combination | Total |
|-------------|-------|
| (1)         | 467   |
| a           | 394   |
| b           | 642   |
| ab          | 571   |

$$\hat{\mu} = \frac{571+394+642+467}{4(6)} = 90.75$$

$$\hat{\alpha}_2 = \frac{571+394-642-467}{4(6)} = -6$$

$$\hat{\beta}_2 = \frac{571-394+642-467}{4(6)} = 14.6666$$

$$(\widehat{\alpha\beta})_{22} = \frac{571-394-642+467}{4(6)} = .08333$$

- $SS_A = 4(6)\hat{\alpha}_2^2 = 864$
- $SS_B = 4(6)\hat{\beta}_2^2 = 5162.67$
- $SS_{AB} = 4(6)\hat{\alpha}\hat{\beta}_{22}^2 = .17$
- Response Surface model

$$y = 90.75 + 14.67x_2 + \epsilon$$

To increase response, model says to increase level. Be careful of interpolation.

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## $2^3$ Factorial Designs

- Eight combinations displayed as cube

| A | B | C | Symbol |
|---|---|---|--------|
| - | - | - | (1)    |
| + | - | - | a      |
| - | + | - | b      |
| + | + | - | ab     |
| - | - | + | c      |
| + | - | + | ac     |
| - | + | + | bc     |
| + | + | + | abc    |

- A main effect is equal to

$$\frac{2\hat{\alpha}_2}{4n} = \frac{(a-(1)) + (ab-b) + (ac-c) + (abc-bc)}{4n}$$

$$= \frac{(a-(1))(b+(1))(c+(1))}{4n}$$

- The AB interaction is equal to

$$\frac{2(\widehat{\alpha\beta})_{22}}{4n} = \frac{((abc-bc) + (ab-b) - (ac-c) + (a-(1)))}{4n}$$

$$= \frac{(a-(1))(b-(1))(c+(1))}{4n}$$

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## 2<sup>3</sup> Factorial

| Contrast Coefficients |   |   |    |   |    |    |     |
|-----------------------|---|---|----|---|----|----|-----|
| Symbol                | A | B | AB | C | AC | BC | ABC |
| (1)                   | - | - | +  | - | +  | +  | -   |
| a                     | + | - | -  | - | +  | +  | +   |
| b                     | - | + | -  | - | +  | -  | +   |
| ab                    | + | + | +  | - | -  | -  | +   |
| c                     | - | - | +  | + | -  | -  | +   |
| ac                    | + | - | -  | + | +  | -  | -   |
| bc                    | - | + | -  | + | -  | +  | -   |
| abc                   | + | + | +  | + | +  | +  | +   |

Contrast =  $\sum c_i y_i = (\pm 1) \pm a \pm b \pm ab \pm c \pm ac \pm bc \pm abc$   
 Parameter = Contrast/8n  
 Effect = Contrast/4n = 2Parameter

- $SS_{\text{Effect}} = (\text{Contrast})^2 / 8n$
- $SS_{\text{Effect}} = 2n(\text{Effect})^2$
- $\text{Var}(\text{Contrast}) = 8n\sigma^2$
- $\text{Var}(\text{Effect}) = \sigma^2 / 2n$

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## General 2<sup>k</sup> Factorial

- Contrast is sum of trt combinations ( $\pm 1$  coeff)

$$\text{Var}(\text{Contrast}) = n2^k \sigma^2$$

- Effect is Contrast divided by  $n2^{k-1}$

$$\text{Var}(\text{Effect}) = \sigma^2 / n2^{k-2}$$

- Can use variances to test significance of effect

Approximate 95% CI for Effect

$$\text{Effect} \pm 1.96 \sqrt{\sigma^2 / n2^{k-2}}$$

- Look at effects whose intervals do not contain 0

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## Yates Algorithm

- For general 2<sup>k</sup> design
  - List data in standard order
  - Create  $k$  new columns in the following manner
  - The first  $2^{k-1}$  entries are obtained by adding adjacent values
  - The last  $2^{k-1}$  entries are obtained by subtracting adjacent values
  - Estimates of effects obtained by dividing  $k$ th column by  $n2^{k-1}$
  - Sum of Squares obtained by squaring  $k$ th column and dividing by  $n2^k$

| Combination | y  | 1 | 2 | 3 | Estimate |
|-------------|----|---|---|---|----------|
| (1)         | 15 |   |   |   |          |
| a           | 10 |   |   |   |          |
| b           | 5  |   |   |   |          |
| ab          | 25 |   |   |   |          |
| c           | 15 |   |   |   |          |
| ac          | 25 |   |   |   |          |
| bc          | 10 |   |   |   |          |
| abc         | 5  |   |   |   |          |

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## Single Replicate In 2<sup>k</sup> Design

- With only one replicate
  - Cannot estimate all interactions and  $\sigma^2$
- Often assume 3 or higher interactions do not occur
- Pool estimates together for error
- Warning: may pool significant interaction
- Create Normal Probability Plot of Effects
- If negligible, effect will be  $N(0, \sigma^2)$
- Will fall along straight line
- Significant effects will not lie along same line
- Use Yates Algorithm or SAS to create plot

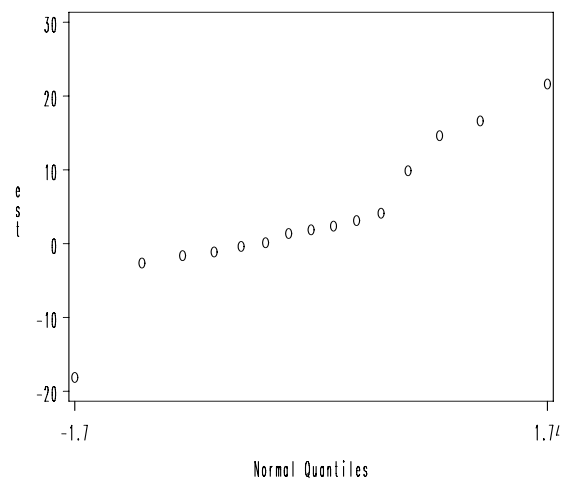
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## Example 6-2

```
data new;
input fact1 fact2 fact3 fact4 est;
cards;
2 1 1 1 21.625
1 2 1 1 3.125
2 2 1 1 0.125
1 1 2 1 9.875
2 1 2 1 -18.125
1 2 2 1 2.375
2 2 2 1 1.875
1 1 1 2 14.625
2 1 1 2 16.625
1 2 1 2 -0.375
2 2 1 2 4.125
1 1 2 2 -1.125
2 1 2 2 -1.625
1 2 2 2 -2.625
2 2 2 2 1.375
;

proc univariate noprint;
qqplot est / normal;
```

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```
data filter;
do D = -1 to 1 by 2; do C = -1 to 1 by 2;
do B = -1 to 1 by 2; do A = -1 to 1 by 2;
input y @@; output;
end; end; end; end;
cards;
45 71 48 65 68 60 80 65 43 100 45 104 75 86 70 96
;

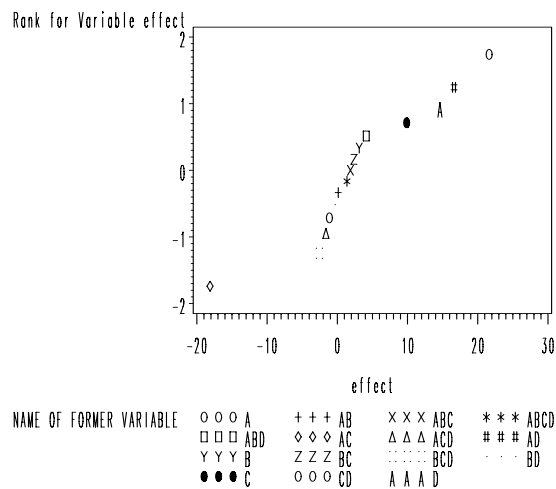
data inter;                                /* Define Interaction Terms */
set filter;
AB=A*B; AC=A*C; AD=A*D; BC=B*C; BD=B*D; CD=C*D; ABC=AB*C; ABD=AB*D;
ACD=AC*D; BCD=BC*D; ABCD=ABC*D;

proc glm data=inter;                        /* GLM Proc to Obtain Effects */
class A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
estimate 'A' A 1 -1; estimate 'AC' AC 1 -1;

proc reg outest=effects data=inter;         /* REG Proc to Obtain Effects */
model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
data effect2; set effects;
drop y intercept _RMSE_;
proc transpose data=effect2 out=effect3;
data effect4; set effect3; effect=col1*2;
proc sort data=effect4; by effect;
proc print data=effect4;
proc rank data=effect4 normal=blom;
var effect; ranks neff;

symbol1 v=circle;
proc gplot;
plot neff*effect=_NAME_;
```

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