Section 7.3 General Factorial and 2^k Factorial Design

Cross Factorial Design with fixed factors

- Model (with three fixed factors, each with multiple levels)

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_l + (\alpha\beta\gamma)_{ij} + (\beta\gamma)_{jk} + (\alpha\gamma)_{il} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl},$$

$$i = 1,\ldots,a;\ j = 1,\ldots,b;\ k = 1,\ldots,c;\ l = 1,\ldots,n$$

where iid errors $\varepsilon_{ijkl} \sim N(0, \sigma^2)$

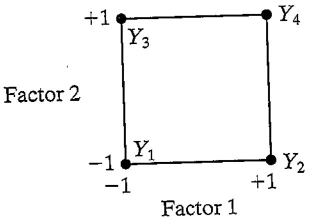
- 2^k Factorial Design with each factor two levels

Table 7.3-1 ANOVA Table for the Factorial Experiment with Three Factors: Fixed Effects

Source	SS	df	MS	F
A	SS_A	$a-1$	MS_A	MS_A / MS_{Error}
B	SS_B	$b-1$	MS_B	MS_B / MS_{Error}
C	SS_C	$c-1$	MS_C	MS_C / MS_{Error}
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	MS_{AB} / MS_{Error}
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	MS_{AC} / MS_{Error}
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	MS_{BC} / MS_{Error}
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	MS_{ABC} / MS_{Error}
Error	SS_{Error}	$abc(n-1)$	MS_{Error}	
Total	SSTO	$abcn-1$		

Table 7.3-2 The 2^2 Factorial Experiment

Run	Design x_1	x_2	x_1x_2	Observation
1	−	−	+	Y_1
2	+	−	−	Y_2
3	−	+	−	Y_3
4	+	+	+	Y_4



The 2^2 Factorial Let us start with $k = 2$ factors and the $2^2 = 4$ factor combinations (low, low), (high, low), (low, high), and (high, high). In coded units, the four runs are $(-1, -1)$, $(1, -1)$, $(-1, 1)$, and $(1, 1)$. We have arranged these runs in Table 7.3-2 in what is called the *standard order*. We start the levels of factor 1 with one minus sign and alternate the signs: −, +, −, +. The levels of factor 2 start with two minus signs,

Table 7.3-3 Example of a 2^2 Factorial Experiment

Run	Design x_1	x_2	x_1x_2	Average Yield	Individual Observations
1	−	−	+	55.0	55.5, 54.5
2	+	−	−	60.6	60.2, 61.0
3	−	+	−	64.2	64.5, 63.9
4	+	+	+	68.2	67.7, 68.7

$$\text{Average} = (55.0 + 60.6 + 64.2 + 68.2)/4 = 62.0$$
$$(1) = (-55.0 + 60.6 - 64.2 + 68.2)/4 = 2.4$$
$$(2) = (-55.0 - 60.6 + 64.2 + 68.2)/4 = 4.2$$
$$(12) = (55.0 - 60.6 - 64.2 + 68.2)/4 = -0.4$$

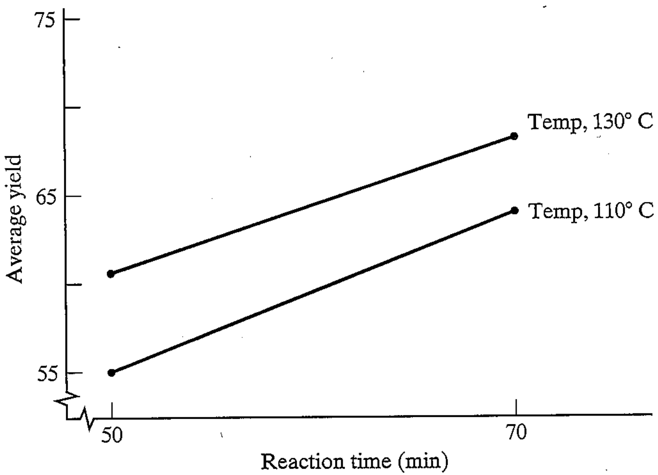
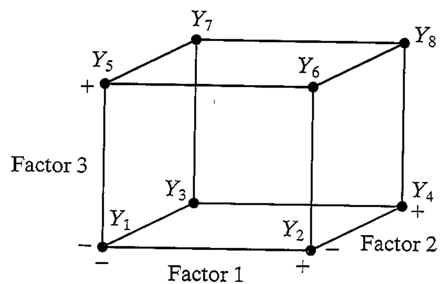


Table 7.3-4 The 2^3 Factorial Experiment

Run	Design x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$	Observation
1	−	−	−	+	+	+	−	Y_1
2	+	−	−	−	−	+	+	Y_2
3	−	+	−	−	+	−	+	Y_3
4	+	+	−	+	−	−	−	Y_4
5	−	−	+	+	−	−	+	Y_5
6	+	−	+	−	+	−	−	Y_6
7	−	+	+	−	−	+	−	Y_7
8	+	+	+	+	+	+	+	Y_8



The main effect of factor i ($i = 1, 2, 3$) is one-half the difference between the averages of the responses at the high and low levels of factor i. Thus, from the cube in Table 7.3-4, we find that

$$(1) = \frac{1}{2}\left(\frac{Y_2 + Y_4 + Y_6 + Y_8}{4} - \frac{Y_1 + Y_3 + Y_5 + Y_7}{4}\right)$$

$$= \frac{-Y_1 + Y_2 - Y_3 + Y_4 - Y_5 + Y_6 - Y_7 + Y_8}{8},$$

Example 7.3-2 As an illustration of a 2^4 factorial, we use the data from an experiment designed to evaluate the effect of laundering on certain fire-retardant treatments for fabrics. [M. G. Natrella, *Experimental Statistics,* National Bureau of Standards Handbook 91 (Washington, D.C.: U.S. Government Printing Office, 1963)]. Factor 1 is the type of fabric (sateen or monk's cloth), factor 2 corresponds to two different fire-retardant

Table 7.3-5 A 2^4 Factorial Experiment

x_1	x_2	x_3	x_4	Y	Effect
−	−	−	−	42	Average = 575/16 = 35.94
+	−	−	−	31	(1) = −129/16 = −8.06
−	+	−	−	45	(2) = 1.56
+	+	−	−	29	(3) = −0.56
−	−	+	−	39	(4) = −0.56
+	−	+	−	28	(12) = −2.19
−	+	+	−	46	(13) = −0.31
+	+	+	−	32	(14) = −1.56
−	−	−	+	40	(23) = 0.81
+	−	−	+	30	(24) = 0.06
−	+	−	+	50	(34) = −0.31
+	+	−	+	25	(123) = 0.31
−	−	+	+	40	(124) = −1.19
+	−	+	+	25	(134) = −0.56
−	+	+	+	50	(234) = −0.44
+	+	+	+	23	(1234) = 0.06

For example, the $x_1x_2x_3$ column reads (written as a row)

$- + + - + - - + - + + - + - - +.$

Thus

$$(123) = \frac{\begin{array}{c}-42 + 31 + 45 - 29 + 39 - 28 - 46 + 32 \\ -40 + 30 + 50 - 25 + 40 - 25 - 50 + 23\end{array}}{16} = 0.31.$$

treatments, factor 3 describes the laundering condition (no laundering, after one laundering), and factor 4 corresponds to two different methods of conducting the flame test. The observations listed in Table 7.3-5 are in inches burned, measured on a standard-size sample fabric after a flame test.

TABLE 8.3-6 **Yates Algorithm for the Data in Example 8.3-2**

x_1	x_2	x_3	x_4	y	*col 1*	*col 2*	*col 3*	*col 4*	Effect
−	−	−	−	42	73	147	292	575	Average = 35.94
+	−	−	−	31	74	145	283	−129	(1) = −8.06
−	+	−	−	45	67	145	−52	25	(2) = 1.56
+	+	−	−	29	78	138	−77	−35	(12) = −2.19
−	−	+	−	39	70	−27	12	−9	(3) = −0.56
+	−	+	−	28	75	−25	13	−5	(13) = −0.31
−	+	+	−	46	65	−35	−8	13	(23) = 0.81
+	+	+	−	32	73	−42	−27	5	(123) = 0.31
−	−	−	+	40	−11	1	−2	−9	(4) = −0.56
+	−	−	+	30	−16	11	−7	−25	(14) = −1.56
−	+	−	+	50	−11	5	2	1	(24) = 0.06
+	+	−	+	25	−14	8	−7	−19	(124) = −1.19
−	−	+	+	40	−10	−5	10	−5	(34) = −0.31
+	−	+	+	25	−25	−3	3	−9	(134) = −0.56
−	+	+	+	50	−15	−15	2	−7	(234) = −0.44
+	+	+	+	23	−27	−12	3	1	(1234) = 0.06

Let us assume that there are n independent observations $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ with variance estimate $S_i^2 = [\sum_{j=1}^{n}(Y_{ij} - \overline{Y}_i)^2]/(n-1)$ at each of the 2^k level combinations, $i = 1, 2, \ldots, 2^k$. The 2^k variance estimates can be pooled to obtain the overall variance estimate

$$S^2 = \frac{1}{2^k}\sum_{i=1}^{2^k} S_i^2 = \frac{1}{(n-1)2^k}\sum_{i=1}^{2^k}\sum_{j=1}^{n}(Y_{ij} - \overline{Y}_i)^2.$$

Because the estimate of the variance of an average $\overline{Y}_i$ at a particular level combination is S^2/n, and because the overall average and each estimated effect can be written as $(1/2^k)\sum_{i=1}^{2^k} c_i \overline{Y}_i$, where the coefficients c_i are either $+1$ or -1, we find that the estimate of the variance of an effect is

$$\text{var(effect)} = \text{var(average)} = \frac{1}{(2^k)^2}\sum_{i=1}^{2^k}\text{var}(\overline{Y}_i) = \frac{S^2}{n2^k}.$$

The estimated effects, together with estimates of their standard deviations which are also known as standard errors, indicate the statistical significance of the various effects.

Example 7.3-3 Using the result derived in Exercise 7.3-1, we find for the data in Table 7.3-3 that

$$s^2 = \frac{1}{4}\left[\frac{(55.5 - 54.5)^2}{2} + \frac{(60.2 - 61.0)^2}{2} + \frac{(64.5 - 63.9)^2}{2} + \frac{(67.7 - 68.7)^2}{2}\right]$$
$$= 0.375,$$

and

$$\text{var(effect)} = \text{var(average)} = \frac{1}{(2)(4)}(0.375) = 0.0469.$$

Hence, the standard error of an estimated effect, as well as of the average, is

$$[\text{var(effect)}]^{1/2} = [\text{var(average)}]^{1/2} = 0.22.$$

Thus, the two-sigma limits around the estimates are 62.0 ± 0.44 for the mean, 2.4 ± 0.44 for the main effect of factor 1, 4.2 ± 0.44 for the main effect of factor 2, and -0.40 ± 0.44 for the two-factor interaction. These intervals are approximate 95 percent confidence intervals and indicate large main effects, but negligible interaction.

Table 7.3-6 Normal Scores of the $m = 2^{4-1} = 15$ Estimated Effects from Example 7.3-2

Identity of Effect	Effect by Magnitude	Rank	$P_i = \frac{i - 0.5}{m}$	z_i
(1)	−8.06	1	0.033	−1.84
(12)	−2.19	2	0.100	−1.28
(14)	−1.56	3	0.167	−0.97
(124)	−1.19	4	0.233	−0.73
(3)	−0.56	6	0.367	−0.34
(4)	−0.56	6	0.367	−0.34
(134)	−0.56	6	0.367	−0.34
(234)	−0.44	8	0.500	0.00
(13)	−0.31	9.5	0.600	0.25
(34)	−0.31	9.5	0.600	0.25
(24)	0.06	11.5	0.733	0.62
(1234)	0.06	11.5	0.733	0.62
(123)	0.31	13	0.833	0.97
(23)	0.81	14	0.900	1.28
(2)	1.56	15	0.967	1.84

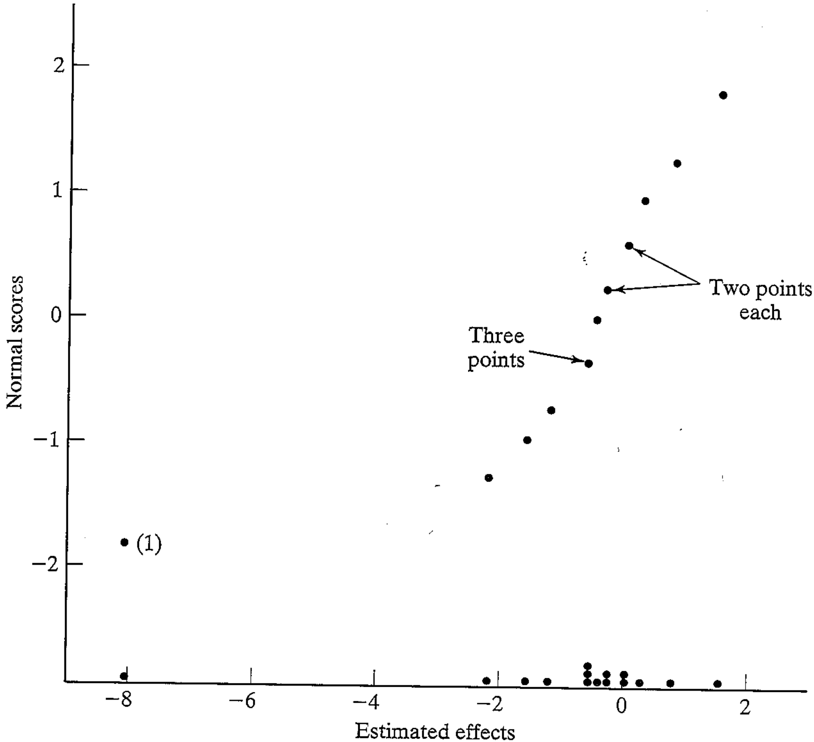


Figure 7.3-1 Dot diagram and normal probability plot of the estimated effects from Example 7.3-2

TABLE 8.4-1 The 2^{3-1} Fractional Factorial with $x_3 = x_1 x_2$

	Design			
Run	x_1	x_2	x_3	Y
1	−	−	+	Y_1
2	+	−	−	Y_2
3	−	+	−	Y_3
4	+	+	+	Y_4

$$L_0 = (Y_1 + Y_2 + Y_3 + Y_4)/4 \rightarrow \mu + (123)$$
$$L_1 = (-Y_1 + Y_2 - Y_3 + Y_4)/4 \rightarrow (1) + (23)$$
$$L_2 = (-Y_1 - Y_2 + Y_3 + Y_4)/4 \rightarrow (2) + (13)$$
$$L_3 = (Y_1 - Y_2 - Y_3 + Y_4)/4 \rightarrow (3) + (12)$$