

Once the samples are observed, the expression

$$\bar{x} - \bar{y} \pm t(\alpha/2; r = n_1 + n_2 - 2) s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

provides a $100(1 - \alpha)$ percent confidence interval for $\mu_1 - \mu_2$.

Remark: The use of a pooled estimate of σ^2 is not recommended if there is suspicion that the variances of the two distributions are not the same. Pooling is especially dangerous in this circumstance if the sample sizes n_1 and n_2 are quite different. In such a case, it is safer to use the approach in Section 4.2-2 and calculate a confidence interval that uses $s_x^2/n_1 + s_y^2/n_2$ as the estimate of $\text{var}(\bar{X} - \bar{Y})$.

Example
4.3-5

An experiment compared the durability of two types of exterior house paint; type *A* was an oil-based paint, while type *B* was a latex paint. A total of $n_1 = 10$ and $n_2 = 12$ wood panels were covered with paints *A* and *B*, respectively. The panels were then exposed to a sequence of tests that involved extreme temperature, light, and moisture conditions. At the end of the experiment, the quality of the painted surface was rated on several characteristics (such as paint loss, luster of painted surface, cracking, and peeling). Scores between 0 (extremely poor) and 100 (excellent) were assigned to the 22 panels. For paint *A* (with $n_1 = 10$), it was found that $\bar{x} = 52$ and $s_x^2 = 400$; for paint *B* (with $n_2 = 12$), it was found that $\bar{y} = 46$ and $s_y^2 = 340$. Because individual measurements are influenced by numerous random factors, it is reasonable to assume that they are normally distributed. We could check this assumption by constructing normal probability plots, one for each group. (See Section 3.6.) A comparison of $s_x^2 = 400$ and $s_y^2 = 340$ shows that the sample variances are quite similar. This provides evidence that the population variances are about equal and justifies the pooling of the individual sample variances. A formal procedure for assessing whether two population variances are the same is given in Section 4.4-1.

Using the preceding theory, we can construct a 90 percent confidence interval for $\mu_1 - \mu_2$. Because $t(\alpha/2 = 0.05; n_1 + n_2 - 2 = 20) = 1.725$, this interval is given by

$$(52 - 46) \pm (1.725) \sqrt{367} \sqrt{\frac{1}{10} + \frac{1}{12}},$$

where the pooled variance is

$$s_p^2 = \frac{(9)(400) + (11)(340)}{10 + 12 - 2} = 367.$$

The 95 percent confidence interval 6 ± 14.1 , or $(-8.1, 20.1)$ includes zero, so there is no evidence in these data that the mean durabilities of the two paints are different. ■

Exercises 4.3

***4.3-1** The mean μ of the tear strength of a certain paper is under consideration. The $n = 22$ determinations (taken at random) yielded $\bar{x} = 2.4$ pounds.

- *(a) If the standard deviation of an individual measurement is known to be $\sigma = 0.2$, find an approximate 95 percent confidence interval for μ .

*(b) If the standard deviation σ is unknown, but the sample standard deviation is $s = 0.2$, determine a 95 percent confidence interval for μ .

***4.3-2** Let W be $\chi^2(12)$.

*(a) Determine the mean and the variance of W .

*(b) Find d_1 , d_2 , d_3 , and d_4 so that $P(W > d_1) = 0.05$, $P(W > d_2) = 0.99$, $P(W > d_3) = 0.005$, and $P(W < d_4) = 0.025$. Use tables or computer software.

***4.3-3** Let T be $t(r = 11)$.

*(a) Determine the mean and the variance of T .

*(b) Determine d_1 so that $P(T > d_1) = 0.05$.

*(c) Determine d_2 so that $P(-d_2 < T < d_2) = 0.95$.

4.3-4 Let μ be the mean mileage of a certain brand of tire. A sample of $n = 14$ tires was taken at random, resulting in $\bar{x} = 32,132$ and $s = 2,596$ miles. Find a 99 percent confidence interval for μ .

4.3-5 Let X be $N(\mu, \sigma^2)$. Then $Z = (X - \mu)/\sigma$ is $N(0, 1)$. Find the distribution function of $W = Z^2$; that is, find $G(w) = P(W \leq w) = P(-\sqrt{w} \leq Z \leq \sqrt{w})$, for $w \geq 0$, such that

$$G(w) = 2 \int_0^{\sqrt{w}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Show that the p.d.f. $g(w) = G'(w)$ of W is $\chi^2(1)$.

Hint: From the fundamental theorem of calculus, we know that

$$\frac{d}{dx} \left[\int_a^{u(x)} g(y) dy \right] = g[u(x)] \frac{d}{dx} u(x).$$

4.3-6 We stated without proof that the quantity $(n - 1)S^2/\sigma^2$ follows a chi-square distribution with $n - 1$ degrees of freedom if our random sample is taken from the $N(\mu, \sigma^2)$ distribution. Convince yourself that this is true by performing the following experiment: Use a computer program that generates values from an $N(\mu = 10, \sigma^2 = 4)$ distribution. From samples of size $n = 4$, calculate \bar{x} , s^2 , and $(n - 1)s^2/\sigma^2 = 3s^2/4$. Repeat the experiment 500 times and construct a normalized relative frequency histogram of $3s^2/4$. [Note: A normalized frequency histogram divides relative frequencies by the widths of the intervals; Minitab calls this a density histogram, and you can get it by clicking on density in the Scale, Y-Scale Type window.] Compare the relative frequencies from the simulations with the p.d.f. of the $\chi^2(3)$ distribution, which is $f(w) = cw^{1/2}e^{-w/2}$, where $c = 0.4$.

***4.3-7** The effectiveness of two methods of teaching statistics is compared. A class of 24 students is randomly divided into two groups and each group is taught according to a different method. The test scores of the two groups at the end of the semester show the following characteristics:

$$n_1 = 13, \quad \bar{x} = 74.5, \quad s_x^2 = 82.6$$

and

we find that the probability that the random interval

$$\frac{Y_1}{n_1} - \frac{Y_2}{n_2} \pm z(\alpha/2) \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

includes $p_1 - p_2$ is about $1 - \alpha$. Performing the experiments, observing y_1 and y_2 , and approximating p_1 and p_2 under the radical sign by y_1/n_1 and y_2/n_2 , respectively, we obtain the approximate $100(1 - \alpha)$ percent confidence interval for $p_1 - p_2$:

$$\frac{y_1}{n_1} - \frac{y_2}{n_2} \pm z(\alpha/2) \sqrt{\frac{(y_1/n_1)(1 - y_1/n_1)}{n_1} + \frac{(y_2/n_2)(1 - y_2/n_2)}{n_2}}.$$

Example 4.4-6

A company suspects that its two major plants produce different proportions of "grade A" items. Samples of sizes $n_1 = n_2 = 300$ were selected from a week's production of the two factories, and $y_1 = 213$ and $y_2 = 189$ items were classified as grade A. Thus, an approximate 95.44 percent confidence interval for $p_1 - p_2$ is

$$0.71 - 0.63 \pm 2 \sqrt{\frac{(0.71)(0.29)}{300} + \frac{(0.63)(0.37)}{300}}$$

or, equivalently, (0.004, 0.156). Because the confidence interval does not include zero, we conclude that the first factory produces, on average, a higher percentage of grade A items than the second.

Exercises 4.4

- 4.4-1** Refer to Exercise 4.2-2. Calculate 95 percent confidence intervals for σ^2 and σ .
- 4.4-2** Refer to Exercises 4.3-7 and 4.3-8, and in each case, compute 90 percent confidence intervals for σ_1^2/σ_2^2 , the ratio of the population variances.
- 4.4-3** One tire manufacturer found that after 5,000 miles, $y = 32$ of $n = 200$ steel-belted tires selected at random were defective. Find an approximate 99 percent confidence interval for p , the proportion of defective tires in the total production.
- *4.4-4** To test two different training methods, 200 workers were divided at random into two groups of 100 each. At the end of the training program, there were $y_1 = 62$ and $y_2 = 74$ successes. Find an approximate 90 percent confidence interval for $p_1 - p_2$, the difference of the true proportions of success.
- *4.4-5** A sample of $n = 21$ observations from a $N(\mu, \sigma^2)$ distribution leads to $\bar{x} = 74.2$ and $s^2 = 562.8$. Determine a 90 percent confidence interval for σ^2 .
- *4.4-6** Let Y be $b(n, p)$. When Y is observed to be y , we want $y/n \pm 0.05$ to be an approximate 95 percent confidence interval for p .
- (a) If we know that p is around $1/4$, how should we choose the sample size n ?
- (b) If we did not have the prior information that p was about $1/4$, what size sample should we take?
- 4.4-7** We want to be 90 percent confident that the difference of two sample propor-

the manufacturer misleads its customers and that the true deflection is, in fact, larger than the one claimed. To see whether this suspicion is justified, the contractor selects $n = 10$ beams at random from his inventory, determines their deflection, and conducts a test of $H_0: \mu = 0.012$ against $H_1: \mu > 0.012$. The 10 measurements are as follows:

0.0132	0.0138	0.0108	0.0126	0.0136
0.0112	0.0124	0.0116	0.0127	0.0131

From these measurements, the contractor calculates the mean $\bar{x} = 0.0125$ and the standard deviation $s = 0.0010$. The test statistic is $(0.0125 - 0.012)/(0.001/\sqrt{10}) = 1.56$, which is smaller than the critical value $t(0.05; n - 1 = 9) = 1.833$. Thus, this sample alone does not provide enough evidence to reject H_0 at significance level $\alpha = 0.05$. Until more measurements are taken, the contractor cannot claim that the company has made a false claim. ■

Exercises 4.5

- 4.5-1** In the introductory illustration of Section 4.5-2, let $n = 300$ (instead of the $n = 100$ used there). Using the Poisson approximation, show that the critical region of $Y \leq 5$ (acceptance region of $Y \geq 6$) has an OC curve such that $OC(0.03) = 0.884$ and $OC(0.01) = 0.084$.
- *4.5-2** Let \bar{X} be the mean of a random sample of size $n = 36$ from $N(\mu, \sigma^2 = 9)$. Our decision rule is to reject $H_0: \mu = 50$ and to accept $H_1: \mu > 50$ if $\bar{X} \geq 50.8$. Determine the $OC(\mu)$ curve and evaluate it at $\mu = 50.0, 50.5, 51.0$, and 51.5 . What is the significance level of the test?
- *4.5-3** Let p be the fraction of engineers who do not understand certain basic statistical concepts. Unfortunately, in the past this number has been high, about $p = 0.73$. A new program to improve the engineers' knowledge of statistical methods has been implemented, and it is expected that under this program p would decrease from 0.73. To test $H_0: p = 0.73$ against $H_1: p < 0.73$, 300 engineers in the new program were tested, and 204 (i.e., 68 percent) did not comprehend certain basic statistical concepts. Compute the probability value required to determine whether these results indicate progress. That is, can we reject H_0 in favor of H_1 ? Use $\alpha = 0.05$.
- 4.5-4** In a certain industry, about 15 percent of the workers showed some signs of ill effects due to radiation. After management had claimed that improvements had been made, 140 workers were tested and 19 experienced some ill effects due to radiation. Does this result support management's claim? Use $\alpha = 0.05$.
- *4.5-5** The mean time required to repair breakdowns of a certain copying machine is 93 minutes. The company which manufactures the machine claims that breakdowns of its new, improved model are easier to fix. To test this claim, $n = 73$ breakdowns of the new model were observed, resulting in a mean repair time of $\bar{x} = 88.8$ minutes and a standard deviation of $s = 26.6$ minutes. Use the significance level $\alpha = 0.05$. What is your conclusion?
- 4.5-6** In an industrial training program, students have been averaging about 65 points on a standardized test. The lecture system was replaced by teaching machines with a lab instructor. There was some doubt as to whether the scores would

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decrease, increase, or stay the same. Hence, $H_0: \mu = 65$ was tested against the two-sided alternative $H_1: \mu \neq 65$. A sample of $n = 50$ students using the teaching machines was tested, resulting in $\bar{x} = 68.2$ and $s = 13.2$. Use a significance level of $\alpha = 0.05$. What is your conclusion?

***4.5-7** Consider a $N(\mu, \sigma^2 = 40)$ distribution. To test $H_0: \mu = 32$ against $H_1: \mu > 32$, we reject H_0 if the sample mean $\bar{X} \geq c$. Find the sample size n and the constant c such that $OC(\mu = 32) = 0.90$ and $OC(\mu = 35) = 0.15$.

***4.5-8** Let Y have a binomial distribution with parameters n and p . In a test of $H_0: p = 0.25$ against $H_1: p < 0.25$, we reject H_0 if $Y/n \leq c$. Find n and c if $OC(p = 0.25) = 0.90$ and $OC(p = 0.20) = 0.05$. State your assumptions.

4.5-9 Let X_1, X_2, \dots, X_{10} be a random sample from a Poisson distribution with mean λ . In a test of $H_0: \lambda = 1.1$ against $H_1: \lambda < 1.1$, we reject H_0 if the sum of the 10 observations, Y , is less than or equal to 8. Using the Poisson table, find the $OC(\lambda)$ curve at $\lambda = 0.5, 0.7, 0.9$, and 1.1 . What is the significance level of the test?

Hint: Use the fact that a sum of n independent Poisson random variables with parameter λ is again Poisson with parameter $n\lambda$.

4.5-10 Recently, a commuter was told that it takes 60 minutes, on average, to travel by car from Philadelphia to Princeton, New Jersey. Anxious to learn whether the 60-minute figure is correct, the commuter takes measurements on $n = 20$ consecutive weekdays and finds that the average time is 68 minutes and the standard deviation is 6 minutes. Is there enough evidence that the time given to the commuter was too low? State the assumptions that you have made in your analysis.

4.5-11 Let \bar{X} be the mean of random sample of size $n = 16$ from a normal distribution with mean μ and standard deviation $\sigma = 8$. To test $H_0: \mu = 35$ against $H_1: \mu > 35$, we reject H_0 if $\bar{X} \geq 36.5$.

(a) Determine the OC curve at $\mu = 35, 36$, and 38.5 .

(b) What is the probability of a type I error?

(c) What is the probability of a type II error at $\mu = 36$?

4.5-12 Sixty-four randomly selected fuses were subjected to a 20 percent overload, and the time to failure was recorded. It was found that the sample average and standard deviation were $\bar{x} = 8.5$ and $s = 2.4$, respectively.

(a) Compute a 99 percent confidence interval for μ , the mean time to failure, under such conditions.

(b) Test $H_0: \mu = 8$ against $H_1: \mu > 8$; use a significance level of $\alpha = 0.05$. What is your conclusion?

4.5-13 Your null hypothesis asserts that a certain game of chance has even odds (i.e., winning and losing are equally likely). Ten people play the game, and you find that 9 of the 10 lose. Calculate the probability value of this sample result, and indicate whether you can reject your null hypothesis (of even odds) in favor of the two-sided alternative at a significance level of 0.05.

4.5-14 The American Association of University Professors claims that the mean income of tenured professors at public universities is \$110,000. Our hypothesis is that the mean salary is lower than \$110,000. To test whether the mean salary is lower than \$110,000 we take a random sample of $n = 36$ professors. Their