Two-Way ANOVA with two random factors.

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \quad \begin{cases} i=1,\cdots,a \\ j=1,\cdots,b \\ k=1,\cdots,n \end{cases}$$

where $\alpha_i \overset{iid}{\sim} N(0,\sigma_\alpha^2)$, $\beta_j \overset{iid}{\sim} N(0,\sigma_\beta^2)$, $(\alpha\beta)_{ij} \overset{iid}{\sim} N(0,\sigma_{\alpha\beta}^2)$

and $\varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma_\varepsilon^2)$

In addition, $\{\alpha_i\}$, $\{\beta_j\}$, $\{(\alpha\beta)_{ij}\}$, $\{\varepsilon_{ijk}\}$ are mutually independent.

① $$Var(\bar{Y}_{i\cdot\cdot}) = Var(\mu + \alpha_i + \bar{\beta} + \overline{(\alpha\beta)}_{i\cdot} + \bar{\varepsilon}_{i\cdot\cdot}) \overset{\text{independence}}{=\!=\!=} \sigma_\alpha^2 + \frac{\sigma_\beta^2}{b} + \frac{\sigma_{\alpha\beta}^2}{b} + \frac{\sigma_\varepsilon^2}{bn}$$

② $$E[SSA] = E\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\right] = bn\cdot E\left[\sum_{i=1}^{a}(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\right]$$

$$\begin{aligned}\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot} &= (\mu+\alpha_i+\bar{\beta}+\overline{(\alpha\beta)}_{i\cdot}+\bar{\varepsilon}_{i\cdot\cdot}) \\ &\quad -(\mu+\bar{\alpha}+\bar{\beta}+\overline{(\alpha\beta)}_{\cdot\cdot}+\bar{\varepsilon}_{\cdot\cdot\cdot}) \\ &= (\alpha_i-\bar{\alpha}) + \left[\overline{(\alpha\beta)}_{i\cdot}-\overline{(\alpha\beta)}_{\cdot\cdot}\right] + (\bar{\varepsilon}_{i\cdot\cdot}-\bar{\varepsilon}_{\cdot\cdot\cdot})^2\end{aligned}$$

$\bar{\alpha}, \bar{\beta} \neq 0$, r.v.

$\alpha_i \overset{iid}{\sim} N(0,\sigma_\alpha^2) \perp\!\!\!\perp (\alpha\beta)_{ij} \overset{iid}{\sim} N(0,\sigma_{\alpha\beta}^2) \perp\!\!\!\perp \varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma_\varepsilon^2)$

$i=1,\cdots,a$ $\quad \alpha_i \overset{iid}{\sim} N(0,\sigma_\alpha^2) \perp\!\!\!\perp \overline{(\alpha\beta)}_{i\cdot} \overset{iid}{\sim} N(0,\frac{\sigma_{\alpha\beta}^2}{b}) \perp\!\!\!\perp \bar{\varepsilon}_{i\cdot\cdot} \overset{iid}{\sim} N(0,\frac{\sigma_\varepsilon^2}{bn})$

$$\begin{aligned}\therefore E\left[\sum_{i=1}^{a}(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\right] &= E\left[\sum_{i=1}^{a}(\alpha_i-\bar{\alpha})^2\right] + E\left[\sum_{i=1}^{a}(\overline{(\alpha\beta)}_{i\cdot\cdot}-\overline{(\alpha\beta)}_{\cdot\cdot})^2\right] + E\left[\sum_{i=1}^{a}(\bar{\varepsilon}_{i\cdot\cdot}-\bar{\varepsilon}_{\cdot\cdot\cdot})^2\right] \\ &= (a-1)\sigma_\alpha^2 + (a-1)\cdot\frac{\sigma_{\alpha\beta}^2}{b} + (a-1)\cdot\frac{\sigma_\varepsilon^2}{bn}\end{aligned}$$

$$\begin{aligned}E[SSA] &= bn\cdot E\left[\sum_{i=1}^{a}(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\right] \\ &= (a-1)\cdot\left[bn\,\sigma_\alpha^2 + n\cdot\sigma_{\alpha\beta}^2 + \sigma_\varepsilon^2\right]\end{aligned}$$

$$\therefore E[MSA] = E\left[\frac{SSA}{a-1}\right] = bn\,\sigma_\alpha^2 + n\,\sigma_{\alpha\beta}^2 + \sigma_\varepsilon^2$$

$$E(SSE) = E\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(Y_{ijk}-\bar{Y}_{ij.})^2\right)$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b} E\left(\sum_{k=1}^{n}(Y_{ijk}-\bar{Y}_{ij.})^2\right)$$

$$Y_{ijk} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\varepsilon_{ijk}$$

$$\bar{Y}_{ij.} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\bar{\varepsilon}_{ij.}$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b} E\left(\sum_{k=1}^{n}(\varepsilon_{ijk}-\bar{\varepsilon}_{ij.})^2\right)$$

$$\varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2) \qquad E\left[\sum_{k=1}^{n}(\varepsilon_{ijk}-\bar{\varepsilon}_{ij.})^2\right] = (n-1)\sigma^2$$

$$\therefore E(SSE) = ab\cdot(n-1)\sigma^2 \quad \Rightarrow \quad E(MSE] = \sigma^2$$

$$SSAB = \sum_i\sum_j\sum_k(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})^2$$

$$\bar{Y}_{ij.} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\bar{\varepsilon}_{ij.}$$

$$\bar{Y}_{i..} = \mu+\alpha_i+\bar{\beta}+\overline{(\alpha\beta)}_{i.}+\bar{\varepsilon}_{i..}$$

$$\bar{Y}_{.j.} = \mu+\bar{\alpha}+\beta_j+\overline{(\alpha\beta)}_{.j}+\bar{\varepsilon}_{.j.}$$

$$\bar{Y}_{...} = \mu+\bar{\alpha}+\bar{\beta}+\overline{(\alpha\beta)}_{..}+\bar{\varepsilon}_{...}$$

$$\therefore E(SSAB) = E\Big(\sum_i\sum_j\sum_k\left((\alpha\beta)_{ij}-\overline{(\alpha\beta)}_{i.}-\overline{(\alpha\beta)}_{.j}+\overline{(\alpha\beta)}_{..}\right)^2$$
$$+\sum_i\sum_j\sum_k\left(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}-\bar{\varepsilon}_{.j.}+\bar{\varepsilon}_{...}\right)^2\Big)$$

$$(*) \quad E\left[\sum_i\sum_j(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}-\bar{\varepsilon}_{.j.}+\bar{\varepsilon}_{...})^2\right]$$

$$= E\sum_i\sum_j(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..})^2 + E\sum_i\sum_j(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})^2 - 2E\left(\sum_i\sum_j(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..})(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})\right)$$

$$(*) = \quad (\mathrm{I}) \quad + \quad (\mathrm{II}) \quad - \quad 2*(\mathrm{III})$$

$$(\mathrm{III}) \quad E\left(\sum_i\sum_j(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..})(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})\right) = E\left[\sum_j(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})\cdot\sum_i(\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..})\right]$$

$$= E\left[\sum_j(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})\cdot a(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})\right]$$

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$$= a\cdot E\left[\sum_{j=1}^{b}(\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{...})^2\right] = a\cdot(b-1)\cdot\frac{\sigma^2}{an} = (b-1)\cdot\frac{\sigma^2}{n}$$

$$(I) = E\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{\varepsilon}_{ij\cdot} - \bar{\varepsilon}_{i\cdot\cdot})^2\right] = E\left(\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{\varepsilon}_{ij\cdot} - \bar{\varepsilon}_{i\cdot\cdot})^2\right) = a\cdot(b-1)\cdot\frac{\sigma^2}{n}$$

as $\bar{\varepsilon}_{ij\cdot} \sim N(0, \frac{\sigma^2}{n})$

$$(II) = E\left(\sum_i\sum_j(\bar{\varepsilon}_{\cdot j\cdot} - \bar{\varepsilon}_{\cdots})^2\right) = a\cdot(b-1)\cdot\frac{\sigma^2}{an} = (b-1)\frac{\sigma^2}{n}$$

$$(*) = (I) + (II) - 2*(III)$$

$$= a(b-1)\frac{\sigma^2}{n} + (b-1)\frac{\sigma^2}{n} - 2(b-1)\frac{\sigma^2}{n} = (a-1)(b-1)\frac{\sigma^2}{n}$$

Similarly we can have

$$E\sum_i\sum_j\left[\left((\alpha\beta)_{ij} - \overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{\cdot j} + \overline{(\alpha\beta)}_{\cdot\cdot}\right)^2\right]$$

$$= (a-1)(b-1)\sigma^2_{\alpha\beta}, \quad \text{as } (\alpha\beta)_{ij} \overset{iid}{\sim} N(0, \sigma^2_{\alpha\beta})$$

$$\therefore E(SSAB) \overset{\sum_{k=1}^{n}}{=} n(a-1)(b-1)\sigma^2_{\alpha\beta} + (a-1)(b-1)\sigma^2$$

$$E[MSAB] = E\left[\frac{SSAB}{(a-1)(b-1)}\right] = \sigma^2 + n\sigma^2_{\alpha\beta}$$

Mixed-effect Model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \qquad \varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

$i = 1, \cdots, a$; $j = 1, \cdots, b$; $k = 1, \cdots, n$

Assumption A - fixed factor, B - random factor

α_i : fixed factor effect at the i-th level

β_j : random effect at the j-th level

$$\sum_{i=1}^{a} \alpha_i = 0 \qquad \beta_j \overset{ind.}{\sim} N(0, \sigma_\beta^2)$$

$$\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0, \quad (\alpha\beta)_{ij} \sim N\left(0, \frac{a-1}{a} \sigma_{\alpha\beta}^2\right)$$

$$Cov\left((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}\right) = \frac{-1}{a} \sigma_{\alpha\beta}^2, \quad i \neq i'$$

and $\{\beta_j\} \perp\!\!\!\perp \{(\alpha\beta)_{ij}\} \perp\!\!\!\perp \{\varepsilon_{ijk}\}$

Mean at the i-level of Factor A:

$$\bar{Y}_{i\cdot\cdot} = \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} (Y_{ijk}) \qquad Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

$$= \mu + \alpha_i + \frac{1}{b} \sum_{j=1}^{b} \beta_j + \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} (\alpha\beta)_{ij} + \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} \varepsilon_{ijk}$$

$$= \mu + \alpha_i + \bar{\beta} + \overline{(\alpha\beta)}_{i\cdot} + \bar{\varepsilon}_{i\cdot\cdot}$$

$$Var(\bar{Y}_{i\cdot\cdot}) = Var(\bar{\beta}) + Var\left(\overline{(\alpha\beta)}_{i\cdot}\right) + Var(\bar{\varepsilon}_{i\cdot\cdot}) \quad \leftarrow \begin{cases} \text{mutually} \\ \text{independence} \end{cases}$$

$$= \frac{1}{b} \sigma_\beta^2 + \frac{1}{b}\left(\frac{a-1}{a} \sigma_{\alpha\beta}^2\right) + \frac{1}{bn} \sigma^2$$

$$= \frac{1}{bn}(\sigma^2 + n\sigma_\beta^2) + \frac{a-1}{ab} \sigma_{\alpha\beta}^2$$

Note that $\quad Cov(\overline{(\alpha\beta)}_{i\cdot}, \overline{(\alpha\beta)}_{i'\cdot}) \qquad \overline{(\alpha\beta)}_{i\cdot} = \frac{1}{b}\sum_{j=1}^{b}(\alpha\beta)_{ij}$

$$= \frac{1}{b^2}\sum_{j=1}^{b}\sum_{j'=1}^{b} Cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j'}) \quad \leftarrow \left((\alpha\beta)_{ij} \perp\!\!\!\perp (\alpha\beta)_{i'j'},\ j\neq j'\right)$$

$$= \frac{1}{b^2}\sum_{j=1}^{b} Cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}) \quad \leftarrow \left(Cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}) = \frac{-1}{a}\sigma^2_{\alpha\beta}\right)$$

$$= \frac{-1}{ab}\sigma^2_{\alpha\beta}$$

$$Var(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}) = Var\left(\overline{(\alpha\beta)}_{i\cdot} - \overline{(\alpha\beta)}_{i'\cdot}\right) + Var(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{i'\cdot\cdot})$$

$$= Var(\overline{(\alpha\beta)}_{i\cdot}) + Var(\overline{(\alpha\beta)}_{i'\cdot}) - 2\,Cov(\overline{(\alpha\beta)}_{i\cdot}, \overline{(\alpha\beta)}_{i'\cdot}) + \frac{2}{bn}\sigma^2$$

$$= \frac{2}{b}\left(\frac{a-1}{a}\right)\sigma^2_{\alpha\beta} - 2\left(\frac{-1}{ab}\sigma^2_{\alpha\beta}\right) + \frac{2}{bn}\sigma^2$$

$$= \frac{2}{b}\sigma^2_{\alpha\beta} + \frac{2}{bn}\sigma^2 = \frac{2}{bn}(\sigma^2 + n\sigma^2_{\alpha\beta})$$

$$E[MSAB] = \sigma^2 + n\sigma^2_{\alpha\beta}$$

Standard error of $\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}$ is $se(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}) = \sqrt{\frac{2}{bn}\cdot MSAB}$

Confidence Interval for $\mu_i - \mu_{i'}$ is

$$(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}) \pm t_{\frac{\alpha}{2}}((a-1)(b-1)) \cdot \sqrt{\frac{2}{bn}\cdot MSAB}$$