①

$$\hat{\mu}_i = \bar{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij} \sim N\left(\mu_i, \frac{\sigma^2}{n_i}\right), \quad Y_{ij} \overset{ind.}{\sim} N(\mu_i, \sigma^2)$$

$Var(\hat{\mu}_i) = \frac{\sigma^2}{n_i}$

$$se^2(\hat{\mu}_i) = \frac{\hat{\sigma}^2}{n_i} = \frac{MSE}{n_i}, \quad \frac{\hat{\mu}_i - \mu_i}{se(\hat{\mu}_i)} \sim t(N-k)$$

$100(1-\alpha)\%$ C.I. for μ_i is:

$$\hat{\mu}_i \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{\mu}_i)$$

or

$$\bar{Y}_i \pm t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{\frac{MSE}{n_i}}$$

② Pairwise comparison $H_0: \mu_i = \mu_j$ vs $H_1: \mu_i \neq \mu_j$.

$\downarrow$

$H_0: \mu_i - \mu_j = 0$ vs $H_1: \mu_i - \mu_j \neq 0$

how many pairs? $\binom{k}{2} = k\cdot(k-1)/2$

hypotheses? $\binom{k}{2}$

$$\hat{\theta}_{ij} = \widehat{\mu_i - \mu_j} = \hat{\mu}_i - \hat{\mu}_j = \bar{Y}_i - \bar{Y}_j$$

$$Var(\hat{\mu}_i - \hat{\mu}_j) \overset{ind.}{=} Var(\hat{\mu}_i) + Var(\hat{\mu}_j) = \frac{\sigma^2}{n_i} + \frac{\sigma^2}{n_j} = \sigma^2\left(\frac{1}{n_i} + \frac{1}{n_j}\right)$$

$$se^2(\hat{\mu}_i - \hat{\mu}_j) = MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right), \quad \frac{\hat{\theta}_{ij}}{se(\hat{\theta}_{ij})} = \frac{\hat{\mu}_i - \hat{\mu}_j - 0}{se(\hat{\mu}_i - \hat{\mu}_j)} \overset{H_0}{\sim} t(N-k)$$

$100(1-\alpha)\%$ C.I. for $(\mu_i - \mu_j)$ is

$$(\hat{\mu}_i - \hat{\mu}_j) \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{\mu}_i - \hat{\mu}_j)$$

i.e.

$$(\bar{Y}_i - \bar{Y}_j) \pm t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Fisher's LSD : $LSD = t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$

Least Significant Difference

$n_1 = \cdots = n_k$, then $LSD = t_{\frac{\alpha}{2}}(N-k)\sqrt{\frac{2MSE}{n}}$

$100(1-\alpha)\%$ C.I. for $\mu_i - \mu_j$ (balanced): $(\bar{Y}_i - \bar{Y}_j) \pm LSD$

③ Multiple Pairwise Comparison (Simultaneous Confidence Interval)

H_0: $\mu_i = \mu_j$ for any pair (i, j) $\qquad H_0: \mu_1 = \cdots = \mu_k \quad \binom{k}{2}$

H_1: at least one pair $\mu_i \neq \mu_j$

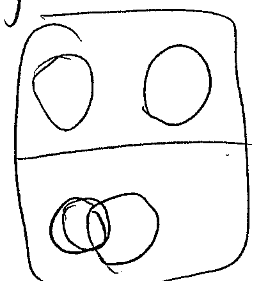
If there are m hypotheses, and significance level α

$\boxed{R_j: \text{~~rejection region~~ } H_{0,j} \text{ for } j\text{-th hypothesis is rejected at level } \alpha_j}$

$$\alpha = P\left(\bigcup_{j=1}^{m} R_j\right)$$

$$\leq \sum_{j=1}^{m} P(R_j) = \sum_{j=1}^{m} \alpha_j \quad R_1, \cdots, R_m \text{ are mutually disjoint}$$

then $\alpha_1 = \cdots = \alpha_m$, then $\alpha_j = \frac{\alpha}{m} \triangleq \alpha^*$.



k tretment levels, total # of pairwise comparisons: $\binom{k}{2} = \frac{k(k-1)}{2} = m$

$$\therefore \quad \alpha^* = \frac{\alpha}{m} = \frac{2\alpha}{k(k-1)}$$

Bonferroni (Simultaneous) C.I. for $(\mu_i - \mu_j)$:

$$(\hat{\mu}_i - \hat{\mu}_j) \pm t_{\frac{\alpha^*}{2}}(N-k) \cdot se(\hat{\mu}_i - \hat{\mu}_j)$$

i.e. $$(\bar{Y}_i - \bar{Y}_j) \pm t_{\frac{\alpha}{k(k-1)}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Conclusion: reject H_0 if there is at least one S.C.I. doesn't include 0

<u>Bonferroni's</u> Extremely general and simple, but often not powerful

<u>Tukey's</u> The best for all-possible pairwise comparisons

Dunnett's Compare "control" to each of the others

Scheffé Unplanned contrast among sets of means

Tukey's Multiple Comparison of Pairwise Difference

H_0: $\mu_i = \mu_j$ for any pair (i, j), $\binom{k}{2}$ subhypotheses

H_1: at least one pair not the same.

For a balanced design $n_1 = \cdots = n_k = n$, then the studentized range statistic can be simplied.

General: $$T_u = \max_{1 \le i, j \le k} \left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} \right| \sim q(k, N-k)$$

$$\left[T_u \xrightarrow[\text{under } H_0: \mu_i = \mu_j]{\text{balanced design}} \left| \frac{\bar{Y}_{max} - \bar{Y}_{min}}{\sqrt{\frac{MSE}{n}}} \right| \right].$$

Then the simultaneous confidence interval can lead to an overall significance level α.

$$P\left((\mu_i - \mu_j) \in (\bar{Y}_i - \bar{Y}_j) \pm q_\alpha(k, N-k) \cdot \sqrt{\frac{MSE}{n}}, \ \forall (i, j) \right)$$

$$\cancel{= P\left((\bar{Y}_i - \bar{Y}_j) \ q_\alpha(k, N-k) \right)} = P\left(\left| (\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j) \right| \le q_\alpha(k, N-k) \sqrt{\frac{MSE}{n}} \ \forall (i, j) \right)$$

$$= P\left(\left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{n}}} \right| \le q_\alpha(k, N-k), \text{ for any } (i, j) \right)$$

$$= P\left(\max_{1 \le i, j \le k} \left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{n}}} \right| \le q_\alpha(k, N-k) \right)$$

$$= \alpha$$

Simultaneous Contrast — Scheffé's Method

$$L = \sum_{i=1}^{K} c_i \mu_i, \qquad \sum_{i=1}^{K} c_i = 0$$

$$\hat{L} = \sum_{i=1}^{K} c_i \bar{Y}_i \quad , \quad se^2(\hat{L}) = \sigma^2 \sum_{i=1}^{K} \frac{c_i^2}{n_i}$$

Simultaneous C.I. for all possible contrast is

$$\hat{L} \pm se(\hat{L}) \cdot \sqrt{(k-1) \cdot F_\alpha(k-1, N-k)}$$

For a general interest in comparing means (not pairwise),

scheffé tends to give narrower confidence limits.

↳ Linear Models

$H_0: C \cdot \mu = 0$

Ellipsoidal confidence region