①

$$\hat{\mu}_i = \bar{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij} \sim N\left(\mu_i, \frac{\sigma^2}{n_i}\right), \qquad Y_{ij} \overset{ind.}{\sim} N(\mu_i, \sigma^2)$$

$Var(\hat{\mu}_i) = \frac{\sigma^2}{n_i}$

$$se^2(\hat{\mu}_i) = \frac{\hat{\sigma}^2}{n_i} = \frac{MSE}{n_i}, \qquad \frac{\hat{\mu}_i - \mu_i}{se(\hat{\mu}_i)} \sim t(N-k)$$

$100(1-\alpha)\%$ C.I. for μ_i is:

$$\hat{\mu}_i \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{\mu}_i)$$

or $$\bar{Y}_i \pm t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{\frac{MSE}{n_i}}$$

② Pairwise comparison $H_0: \mu_i = \mu_j$ vs $H_1: \mu_i \neq \mu_j$.

$\downarrow$

how many pairs? $\binom{k}{2} = k\cdot(k-1)/2$ $\qquad H_0: \mu_i - \mu_j = 0$

hypotheses? $\binom{k}{2}$ $\qquad$ vs $H_1: \mu_i - \mu_j \neq 0$

$$\hat{\theta}_{ij} = \widehat{\mu_i - \mu_j} = \hat{\mu}_i - \hat{\mu}_j = \bar{Y}_i - \bar{Y}_j$$

$$Var(\hat{\mu}_i - \hat{\mu}_j) \overset{ind.}{=} Var(\hat{\mu}_i) + Var(\hat{\mu}_j) = \frac{\sigma^2}{n_i} + \frac{\sigma^2}{n_j} = \sigma^2\left(\frac{1}{n_i} + \frac{1}{n_j}\right)$$

$$se^2(\hat{\mu}_i - \hat{\mu}_j) = MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right), \quad \frac{\hat{\theta}_{ij}}{se(\hat{\theta}_{ij})} = \frac{\hat{\mu}_i - \hat{\mu}_j - 0}{se(\hat{\mu}_i - \hat{\mu}_j)} \overset{H_0}{\sim} t(N-k)$$

$100(1-\alpha)\%$ C.I. for $(\mu_i - \mu_j)$ is

$$(\hat{\mu}_i - \hat{\mu}_j) \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{\mu}_i - \hat{\mu}_j)$$

i.e. $$(\bar{Y}_i - \bar{Y}_j) \pm t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Fisher's LSD : $\quad LSD = t_{\frac{\alpha}{2}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$

Least Significant Difference

$n_1 = \cdots = n_k$, then $LSD = t_{\frac{\alpha}{2}}(N-k)\sqrt{\frac{2MSE}{n}}$

$100(1-\alpha)\%$ C.I. for $\mu_i - \mu_j$ (balanced): $(\bar{Y}_i - \bar{Y}_j) \pm LSD$

Estimate $D = \mu_3 - \mu_4$ (with/without cartoons, 5 colors) $N=19$, $k=4$

$\hat{D} = \bar{Y}_{3.} - \bar{Y}_{4.} = 19.5 - 27.2 = -7.7$

$se^2(\hat{D}) = MSE \cdot \left(\frac{1}{n_3} + \frac{1}{n_4}\right) = 10.55\left(\frac{1}{4} + \frac{1}{5}\right) = 4.748 \xrightarrow{\sqrt{\cdot}} se(\hat{D}) = 2.179$

95% C.I. for $\mu_3 - \mu_4$: $\hat{D} \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{D})$

$-7.7 \pm 2.131 \cdot (2.179)$. $N-k=15$

i.e. $-12.3 < \mu_3 - \mu_4 < -3.1$

$0 \notin (-12.3, -3.1) \rightarrow$ cartoon has an effect

Test $H_0: \mu_3 = \mu_4$ $H_1: \mu_3 \neq \mu_4$

$t = \frac{\hat{D}}{se(\hat{D})} \sim t(15)$ under H_0

$t_0 = \frac{-7.7}{2.179} = -3.53$, $|t_0| > 2.131 \rightarrow$ in favor of H_1

Pvalue $= 2 \times P\{t(15) > 3.53\} = 0.003$

* Inference for contrast $L = \sum_{i=1}^{k} c_i \mu_i$ subject to $\sum_i c_i = 0$, $\sum_i (c_i)_+ = 1$, $\sum_i (c_i)_- = 1$

$Var(\hat{L}) = Var\left(\sum_{i=1}^{k} c_i \hat{\mu}_i\right) = \sum_{i=1}^{k} \frac{c_i^2}{n_i} \sigma^2$, $\hat{L} \sim N\left(L, \sigma^2 \cdot \sum_{i=1}^{n} \frac{c_i^2}{n_i}\right)$

$\therefore se^2(\hat{L}) = MSE \cdot \sum_{i=1}^{k} \frac{c_i^2}{n_i}$

$\Rightarrow \hat{L} - L \big/ se(\hat{L}) \sim t(N-k)$

$100(1-\alpha)\%$ C.I. for L: $\hat{L} \pm t_{\frac{\alpha}{2}}(N-k) \cdot se(\hat{L})$ — individual C.I.

Example: $\underbrace{\frac{\mu_1}{2} + \frac{\mu_2}{2} - \frac{\mu_3}{2} - \frac{\mu_4}{2} = 0}_{\text{color effect}}$ $\underbrace{\frac{\mu_1}{2} - \frac{\mu_2}{2} + \frac{\mu_3}{2} - \frac{\mu_4}{2} = 0}_{\text{carton effect}}$

③ Multiple Pairwise Comparison (Simultaneous Confidence Interval)

H_0: $\mu_i = \mu_j$ for any pair (i, j) $\qquad H_0: \mu_1 = \cdots = \mu_k \quad \binom{k}{2}$

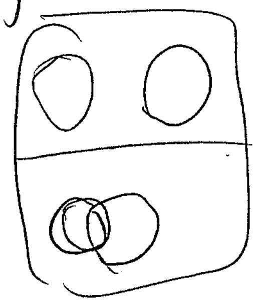
H_1: at least one pair $\mu_i \neq \mu_j$

If there are m hypotheses, and significance level α

$\boxed{R_j: \text{~~rejection region~~ } H_{0,j} \text{ for j-th hypothesis is rejected at level } \alpha_j}$

$$\alpha = P\left(\bigcup_{j=1}^{m} R_j\right)$$

$$\leq \sum_{j=1}^{m} P(R_j) = \sum_{j=1}^{m} \alpha_j \quad R_1, \cdots, R_m \text{ are mutually disjoint}$$



then $\alpha_1 = \cdots = \alpha_m$, then $\alpha_j = \frac{\alpha}{m} \triangleq \alpha^*$.

k tretment levels, total # of pairwise comparisons: $\binom{k}{2} = \frac{k(k-1)}{2} = m$

$$\therefore \quad \alpha^* = \frac{\alpha}{m} = \frac{2\alpha}{k(k-1)}$$

Bonferroni Simultaneous C.I. for $(\mu_i - \mu_j)$:

$$(\hat{\mu}_i - \hat{\mu}_j) \pm t_{\frac{\alpha^*}{2}}(N-k) \cdot se(\hat{\mu}_i - \hat{\mu}_j)$$

i.e. $$(\bar{Y}_i - \bar{Y}_j) \pm t_{\frac{\alpha}{k(k-1)}}(N-k) \cdot \sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Conclusion: reject H_0 if there is at least one S.C.I. doesn't include 0

<u>Bonferroni's</u> Extremely general and simple, but often not powerful

<u>Tukey's</u> the best for all-possible pairwise comparisons

Dunnett's Compare "control" to each of the others

Scheffé Unplanned contrast among sets of means

Tukey's Multiple Comparison of Pairwise Difference

H_0: $\mu_i = \mu_j$ for any pair (i, j), $\binom{k}{2}$ subhypotheses

H_1: at least one pair not the same.

For a balanced design $n_1 = \cdots = n_k = n$, then the studentized range statistic can be simplied.

General: $$T_u = \max_{1 \le i, j \le k} \left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} \right| \sim q(k, N-k)$$

$$\left[T_u \xrightarrow[\text{under } H_0: \mu_i = \mu_j]{\text{balanced design}} \left| \frac{\bar{Y}_{max} - \bar{Y}_{min}}{\sqrt{\frac{MSE}{n}}} \right| \right]$$

Then the simultaneous confidence interval can lead to an overall significance level α.

$$P\left((\mu_i - \mu_j) \in (\bar{Y}_i - \bar{Y}_j) \pm q_\alpha(k, N-k) \cdot \sqrt{\frac{MSE}{n}}, \ \forall (i,j) \right)$$

$$\cancel{= P\left((\bar{Y}_i - \bar{Y}_j) = q_\alpha(k, N-k)\right)} = P\left(\left| (\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j) \right| \le q_\alpha(k, N-k) \sqrt{\frac{MSE}{n}} \ \forall (i,j) \right)$$

$$= P\left(\left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{n}}} \right| \le q_\alpha(k, N-k), \text{ for any } (i,j) \right)$$

$$= P\left(\max_{1 \le i, j \le k} \left| \frac{(\bar{Y}_i - \mu_i) - (\bar{Y}_j - \mu_j)}{\sqrt{\frac{MSE}{n}}} \right| \le q_\alpha(k, N-k) \right)$$

$$= \alpha$$

Simultaneous Contrast — Scheffé's Method

$$L = \sum_{i=1}^{K} c_i \mu_i, \qquad \sum_{i=1}^{K} c_i = 0$$

$$\hat{L} = \sum_{i=1}^{K} c_i \bar{Y}_i, \qquad se^2(\hat{L}) = \sigma^2 \sum_{i=1}^{K} \frac{c_i^2}{n_i}$$

Simultaneous C.I. for all possible contrast is

$$\hat{L} \pm se(\hat{L}) \cdot \sqrt{(K-1) \cdot F_\alpha(K-1, N-K)}$$

For a general interest in comparing means (not pairwise),

scheffé tends to give narrower confidence limits.

↳ Linear Models

$H_0: C \cdot \mu = 0$

Ellipsoidal confidence region

Random-effects model (ANOVA)
(Variance Component)

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad i=1,\cdots,k, \quad j=1,\cdots,n_i\ (n)$$

where μ: overall mean

τ_i: treatment effect, $\tau_i \overset{iid}{\sim} N(0, \sigma_\tau^2)$

ε_{ij}: error at j-th observation of i-th factor level
$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$

k: # of treatment (factor) levels

n_i: # of replication at i-th treatment level

In addition, assume that all τ_i and ε_{ij} are independent.

Note that $\bar{\tau} = \frac{1}{k}\sum_{i=1}^{k} \tau_i$ (balanced case)

is a random variable, $\bar{\tau} \sim N\left(0, \frac{\sigma_\tau^2}{k}\right)$

While in fixed effect, $\bar{\tau}$ is a constant 0, i.e.

$$\bar{\tau} = \frac{1}{k}\sum_{i=1}^{k} \tau_i \equiv 0$$

$$Var(Y_{ij}) = \sigma^2 + \sigma_\tau^2$$

$$\begin{aligned}Cov(Y_{ij}, Y_{ij'}) &= Cov(\mu + \tau_i + \varepsilon_{ij}, \ \mu + \tau_i + \varepsilon_{ij'}) \\ &= Cov(\tau_i, \tau_i + \varepsilon_{ij'}) + Cov(\varepsilon_{ij}, \tau_i + \varepsilon_{ij'}) \\ &= Cov(\tau_i, \tau_i) + Cov(\tau_i, \varepsilon_{ij'}) + Cov(\varepsilon_{ij}, \tau_i) \\ &\qquad + Cov(\varepsilon_{ij}, \varepsilon_{ij'}) \\ &= \sigma_\tau^2\end{aligned}$$

$$Cov(Y_{i'j'}, Y_{ij}) = Cov(\mu + \tau_{i'} + \varepsilon_{i'j'}, \ \mu + \tau_i + \varepsilon_{ij}) = 0$$

$$\rho(Y_{ij}, Y_{ij'}) = \frac{Cov(Y_{ij}, Y_{ij'})}{\sqrt{Var(Y_{ij}) \cdot Var(Y_{ij'})}} = \frac{\sigma_\tau^2}{\sigma^2 + \sigma_\tau^2}$$

$\tau_i \overset{iid}{\sim} N(0, \sigma_\tau^2)$, $i=1,\cdots,K$, then $S_\tau^2 = \frac{1}{K-1}\sum_{i=1}^{K}(\tau_i - \bar\tau)^2$, $\bar\tau = \frac{1}{K}\sum_{i=1}^{K}\tau_i$

Student's Theorem, $\frac{(K-1)S_\tau^2}{\sigma_\tau^2} \sim \chi^2(K-1)$. $E[S_\tau^2] = \sigma_\tau^2$

and $E\left(\sum_{i=1}^{K}(\tau_i - \bar\tau)^2\right) = (K-1)\sigma_\tau^2$

$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$, $\bar\varepsilon_i = \frac{1}{n}\sum_{j=1}^{n}\varepsilon_{ij} \overset{iid}{\sim} N\left(0, \frac{\sigma^2}{n}\right)$, $i=1,\cdots,K$.

$\bar\varepsilon = \frac{1}{K}\sum_{i=1}^{K}\bar\varepsilon_i$, then $E\left[\sum_{i=1}^{K}(\bar\varepsilon_i - \bar\varepsilon)^2\right] = (K-1)\cdot\frac{\sigma^2}{n}$

$$\begin{aligned}\therefore E[SSTR] &= E\left[\sum_{i=1}^{K} n(\bar Y_{i\cdot} - \bar Y)^2\right] \\ &= n \cdot E\left[\sum_i(\tau_i - \bar\tau)^2 + \sum_i(\bar\varepsilon_i - \bar\varepsilon)^2\right] \\ &= n\left[(K-1)\sigma_\tau^2 + (K-1)\cdot\frac{\sigma^2}{n}\right] \\ &= n(K-1)\sigma_\tau^2 + (K-1)\cdot\sigma^2\end{aligned}$$

$E[MSTR] = n\sigma_\tau^2 + \sigma^2$, $\quad E\left[\frac{MSTR - MSE}{n}\right] = \sigma_\tau^2$