

Residuals and Fitted Values

$$\text{Fitted Values: } \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i, \quad i=1, \dots, n$$

$$\text{where } \hat{\beta}_1 = s_{xy}/s_{xx}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\text{Residuals: } e_i = y_i - \hat{y}_i, \quad i=1, \dots, n$$

$$\text{Two restrictions: (1) } \sum_i e_i = 0, \quad (2) \sum_i e_i x_i = 0$$

$$\begin{aligned} (1) \sum_i e_i &= \sum_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) & \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ &= \sum_i [(y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{x})] \\ &= \sum_i (y_i - \bar{y}) - \hat{\beta}_1 \sum_i (x_i - \bar{x}) = 0 - \hat{\beta}_1 \cdot 0 = 0 \end{aligned}$$

$$\begin{aligned} (2) \sum_i e_i x_i &= \sum_i [(y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{x})] \cdot x_i \\ &= \sum_i (y_i - \bar{y}) \cdot x_i - \hat{\beta}_1 \sum_i (x_i - \bar{x}) \cdot x_i \\ &= \sum_i (y_i - \bar{y})(x_i - \bar{x}) - \hat{\beta}_1 \sum_i (x_i - \bar{x})(x_i - \bar{x}) \\ &= s_{xy} - \left(\frac{s_{xy}}{s_{xx}} \right) \cdot s_{xx} = 0 \end{aligned}$$

$$e = (e_1, \dots, e_n)', \quad x = (x_1, \dots, x_n)', \quad e'x = 0 \Leftrightarrow e \perp x, \text{ orthogonal}$$

It is easy to show that

$$\begin{aligned} \sum_i e_i (\hat{y}_i - \bar{y}) &= \sum_i e_i (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y}) \\ &= \sum_i e_i (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y}) = \sum_i e_i \cdot \hat{\beta}_1 (x_i - \bar{x}) \\ &= \hat{\beta}_1 \cdot \sum_i e_i (x_i - \bar{x}) = \hat{\beta}_1 \cdot \sum_i e_i x_i = 0 \end{aligned}$$

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Cochran's Theorem - Decomposition of χ^2 -distribution

(1) If $X_1 \sim \chi^2(r_1)$, $X_1 + X_2 \sim \chi^2(r)$, X_1 is independent of X_2
then $X_2 \sim \chi^2(r - r_1)$.

(2) If $X_1 \sim \chi^2(r_1)$, $X_2 \sim \chi^2(r_2)$, and $X_1 + X_2 \sim \chi^2(r_1 + r_2)$
then X_1 and X_2 are independent.

Decomposition of Sum Square: $SST = SSR + SSE$

$$SSR = \sum_i (x_i - \bar{x})^2 \cdot \hat{\beta}_1^2 = S_{XX} \cdot \hat{\beta}_1^2$$

$$(1) \hat{\beta}_1 = \frac{S_{XY}}{S_{XX}} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{S_{XX}} = \frac{\sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{XX}} (y_i - \bar{y})}{1} = \sum_{i=1}^n c_i (y_i - \bar{y})$$

where $c_i = \frac{x_i - \bar{x}}{S_{XX}} = \frac{x_i - \bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2}$ with $\sum_{i=1}^n c_i = 0$, $\sum_{i=1}^n c_i x_i = \sum_{i=1}^n c_i (x_i - \bar{x}) = 1$
 $\sum_{i=1}^n c_i^2 = S_{XX}^{-1} = \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right]^{-1}$

$$\left. \begin{array}{l} \hat{\beta}_1 = \sum_{i=1}^n c_i y_i \\ y_i \stackrel{\text{ind.}}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2) \end{array} \right\} \Rightarrow \hat{\beta}_1 \sim N\left(\sum_{i=1}^n c_i \beta_0 + \sum_{i=1}^n c_i x_i \beta_1, \sum_{i=1}^n c_i^2 \sigma^2\right) \\ \sim N\left(\beta_1, \frac{\sigma^2}{S_{XX}}\right)$$

$$(2) \therefore \frac{SSR}{\sigma^2} = \frac{\hat{\beta}_1^2 \cdot S_{XX}}{\sigma^2} = \frac{(\hat{\beta}_1 - 0)^2}{\sigma^2 / S_{XX}} \stackrel{H_0: \beta_1 = 0}{\sim} \chi^2(1)$$

In addition,
 $SSE \perp SSR$
as $\text{Cov}(e, \beta) = 0$

$$\frac{SST}{\sigma^2} = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2} = \frac{(n-1) \cdot S_y^2}{\sigma^2} \sim \chi^2(n-1)$$

$$SSE = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n e_i^2 \Rightarrow \frac{SSE}{\sigma^2} \sim \chi^2(n-2)$$

Cochran's Theorem

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \sim N(\mu_{\beta_0}, \sigma_{\beta_0}^2)$$

$$\mu_{\beta_0} = E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{x})$$

$$y_i \stackrel{\text{ind.}}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$= \frac{1}{n} \sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \hat{\beta}_1 \cdot \bar{x} = \beta_0$$

$$\sigma_{\beta_0}^2 = \text{Var}(\hat{\beta}_0) = \text{Var}(\bar{y} - \hat{\beta}_1 \bar{x})$$

$$\left[\begin{aligned} \bar{y} - \hat{\beta}_1 \bar{x} &= \frac{1}{n} \sum_{i=1}^n y_i - \left(\sum_{i=1}^n c_i y_i \right) \bar{x} \\ &= \sum_{i=1}^n \left(\frac{1}{n} - c_i \bar{x} \right) y_i \end{aligned} \right]$$

$$\hat{\beta}_1 = \sum_i c_i y_i, \quad c_i = \frac{x_i - \bar{x}}{S_{XX}}$$

$$\therefore \sigma_{\beta_0}^2 = \text{Var} \left(\sum_{i=1}^n \left(\frac{1}{n} - c_i \bar{x} \right) y_i \right)$$

$$= \sigma^2 \sum_{i=1}^n \left(\frac{1}{n} - c_i \bar{x} \right)^2 = \sigma^2 \left(\frac{1}{n} - \frac{2}{n} \bar{x} \sum_i c_i + \bar{x}^2 \sum_i c_i^2 \right)$$

$$= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) = \frac{\sigma^2 \sum_i x_i^2}{S_{XX}}$$

$$\left[\sum_i (x_i - \bar{x})^2 + n \bar{x}^2 = \sum_i x_i^2 \right]$$

$$\therefore \hat{\beta}_0 \sim N \left(\beta_0, \frac{\sigma^2 \sum_i x_i^2}{S_{XX}} \right)$$

Note that: $\hat{\beta}_1 \sim N \left(\beta_1, \frac{\sigma^2}{S_{XX}} \right)$

$$E(\text{MSE}) = E \left[\frac{\text{SSE}}{n-2} \right] = \sigma^2$$

$$\hat{\sigma}^2 = \text{MSE}$$

$$\therefore \text{se}(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{S_{XX}}} = \sqrt{\frac{\text{MSE}}{S_{XX}}}$$

$$\text{se}(\hat{\beta}_0) = \sqrt{\frac{\text{MSE} \cdot \sum_i x_i^2}{S_{XX}}}$$

Sampling Distribution of $\hat{\beta}_1$

$$\frac{\hat{\beta}_1 - \beta_1}{se(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{Var(\hat{\beta}_1)}} \stackrel{\sim N(0,1)}{=} \frac{Z_{\beta_1}}{\sqrt{\frac{MSE/S_{XX}}{\sigma^2/S_{XX}}}} = \frac{Z_{\beta_1}}{\sqrt{\frac{MSE}{\sigma^2}}}$$

$$\frac{MSE}{\sigma^2} = \frac{SSE}{\sigma^2 / (n-2)} \sim \chi^2_{(n-2)} / (n-2)$$

$$\therefore \frac{\hat{\beta}_1 - \beta_1}{se(\hat{\beta}_1)} = \frac{Z_{\beta_1}}{\sqrt{\chi^2_{(n-2)} / (n-2)}}, \quad \hat{\beta}_1 \perp SSE$$

$$\sim t_{(n-2)}$$

100(1- α)% C.I. for β_1 :

$$\hat{\beta}_1 \pm t_{\frac{\alpha}{2}, (n-2)} \cdot se(\hat{\beta}_1)$$

Confidence interval for β_0 : $\hat{\beta}_0 \pm t_{\frac{\alpha}{2}, (n-2)} \cdot se(\hat{\beta}_0)$, $se(\hat{\beta}_0) = \sqrt{\frac{MSE \cdot \sum_i x_i^2}{n S_{XX}}}$

C.I. for β_1 : $\hat{\beta}_1 \pm t_{\frac{\alpha}{2}, (n-2)} \cdot se(\hat{\beta}_1)$, $se(\hat{\beta}_1) = \sqrt{\frac{MSE}{S_{XX}}}$

* Confidence interval for $\beta_0 + \beta_1 x_i$ (Mean Response).

(1) LS estimators: $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$, $\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}}$

Fitted values: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$, $i=1, \dots, n$

Mean Response $E Y_i = \beta_0 + \beta_1 x_i$, $i=1, \dots, n$, $Y_i \stackrel{\text{ind.}}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2)$

Note that $E(\hat{Y}_i) = \beta_0 + \beta_1 x_i = E(Y_i)$,

(2) $Var(\hat{Y}_i) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_i)$
 $= Var(\hat{\beta}_0) + x_i^2 \cdot Var(\hat{\beta}_1) + 2x_i \cdot Cov(\hat{\beta}_0, \hat{\beta}_1)$

where $Cov(\hat{\beta}_0, \hat{\beta}_1) = Cov(\bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1)$
 $= Cov(\bar{y}, \hat{\beta}_1) - \bar{x} \cdot Var(\hat{\beta}_1) = -\bar{x} Var(\hat{\beta}_1)$

$$\left[\begin{aligned} \hat{\beta}_1 &= \sum_i c_i y_i = \sum_i \frac{x_i - \bar{x}}{S_{XX}} y_i, \quad \bar{y} = \sum_i \left(\frac{1}{n}\right) y_i \\ Cov(\bar{y}, \hat{\beta}_1) &= Cov\left(\sum_{i=1}^n \left(\frac{1}{n}\right) y_i, \sum_{i=1}^n c_i y_i\right) \\ &\quad y_1, \dots, y_n \text{ are independent} \\ Cov(\bar{y}, \hat{\beta}_1) &= \sum_{i=1}^n \frac{c_i}{n} Var(y_i) = \frac{1}{n} \left(\sum_i c_i\right) \sigma^2 = 0 \end{aligned} \right.$$

$\Rightarrow Var(\hat{Y}_i) = Var(\hat{\beta}_0) + x_i^2 \cdot Var(\hat{\beta}_1) - 2x_i \cdot \bar{x} \cdot Var(\hat{\beta}_1)$

$= \frac{\sigma^2}{n S_{XX}} \left(\sum_i x_i^2 + n x_i^2 - 2x_i \cdot \sum_i x_i \right)$

$= \frac{\sigma^2}{n S_{XX}} \left[\sum_i (x_i - \bar{x})^2 + n(x_i - \bar{x})^2 \right] = \sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}} \right)$

(3) Standard Error of $\hat{Y}_i$: $se(\hat{Y}_i)$

$$se^2(\hat{Y}_i) = \hat{\sigma}^2 \cdot \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right) = MSE \cdot \left[\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right]$$

where $MSE = \frac{SSE}{n-2}$ is unbiased for σ^2 .

(4) Sampling Distribution of $\hat{Y}_i$:

$$\begin{aligned} \frac{\hat{Y}_i - E(\hat{Y}_i)}{se(\hat{Y}_i)} &= \frac{\hat{Y}_i - (\beta_0 + \beta_1 x_i)}{\sqrt{MSE \cdot \left[\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right]}} \\ &= \frac{[\hat{Y}_i - (\beta_0 + \beta_1 x_i)] / \sqrt{\sigma^2 \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right)}}{\sqrt{\frac{SSE}{\sigma^2} / (n-2)}} \end{aligned}$$

$$\stackrel{D}{=} \frac{Z \sim N(0,1)}{\sqrt{\chi^2_{(n-2)} / (n-2)}} \quad \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \perp SSE$$

$$\sim t_{(n-2)}$$

100(1- α)% Confidence interval for $(\beta_0 + \beta_1 x_i)$ is

$$\hat{Y}_i \pm t_{\frac{\alpha}{2}, (n-2)} \cdot se(\hat{Y}_i)$$

$$\text{where } \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i, \quad se(\hat{Y}_i) = \sqrt{MSE \left[\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{xx}} \right]}$$

Sampling Distribution of Autocorrelation

Residuals of simple linear regression: $e_1, \dots, e_n$. Suppose data is collected over time $t = 1, \dots, n$ and normally distributed with constant variance σ^2 , i.e. $e_1, \dots, e_n \sim i.i.d.N(0, 1)$

Lag 1 autocorrelation

$$r_1 = \frac{\sum_{t=2}^n e_{t-1}e_t}{\sum_{t=1}^n e_t^2}$$

Under the hypothesised independence among the residuals, the numerator has mean $E\left[\sum_{t=2}^n e_{t-1}e_t\right] = 0$ and variance

$$\begin{aligned} \text{var}\left[\sum_{t=2}^n e_{t-1}e_t\right] &= E\left[\sum_{t=2}^n e_{t-1}e_t\right]^2 \\ &= E\left[\sum_{s=2}^n \sum_{t=2}^n e_{s-1}e_s e_{t-1}e_t\right] = E\sum_{t=2}^n e_{t-1}^2 e_t^2 = (n-1)\sigma^4. \end{aligned}$$

For large n , we can show that $\sum_{t=2}^n e_{t-1}e_t$ is asymptotic normal $N(0, (n-1)\sigma^4)$, or $\sigma^{-2}(n-1)^{-1/2} \sum_{t=2}^n e_{t-1}e_t \sim AN(0, 1)$.

Denominator $SSE = \sum_{t=1}^n e_t^2$. And $MSE = SSE/(n-2)$ is an estimate for σ^2 . Hence

$$r_1 = \frac{\sum_{t=2}^n e_{t-1}e_t}{\sum_{t=1}^n e_t^2} \sim AN\left(0, \frac{1}{n}\right).$$

The 95% confidence interval of autocorrelation coefficient could be approximated by

$$r_1 \pm 2\sqrt{\frac{1}{n}}.$$