Chapter 4

Part II. Hypothesis Testing

Using Statistics

- A hypothesis is a statement or assertion about the state of nature (about the true value of an unknown population parameter):
  - ✓ The accused is innocent
  - ✓ $\mu = 100$
- Every hypothesis implies its contradiction or alternative:
  - ✓ The accused is guilty
  - ✓ $\mu \neq 100$
- A hypothesis is either true or false, and you may fail to reject it or you may reject it on the basis of information:
  - ✓ Trial testimony and evidence
  - ✓ Sample data

Statistical Hypothesis Testing

- A **null hypothesis**, denoted by H_0, is an assertion about one or more population parameters. This is the assertion we hold to be true until we have sufficient statistical evidence to conclude otherwise.
  - ✓ H_0: $\mu = 100$
- The **alternative hypothesis**, denoted by H_1, is the assertion of all situations *not* covered by the null hypothesis.
  - ✓ H_1: $\mu \neq 100$ (two-tailed test)

- H_0 and H_1 are:
  - ✓ Mutually exclusive
    - – Only one can be true.
  - ✓ Exhaustive
    - – Together they cover *all* possibilities, so one or the other *must* be true.

Hypothesis about other Parameters

- Hypotheses about other parameters such as population proportions and and population variances are also possible. For example

  ✓ H_0: $p \geq 40\%$
  ✓ H_1: $p < 40\%$ (left-tailed test)

  ✓ H_0: $\sigma^2 \leq 50$
  ✓ H_1: $\sigma^2 > 50$ (right-tailed test)

The Concepts of Hypothesis Testing

- A **test statistic** is a sample statistic computed from sample data. The value of the test statistic is used in determining whether or not we may reject the null hypothesis.
- The **decision rule** of a statistical hypothesis test is a rule that specifies the conditions under which the null hypothesis may be rejected.

Consider H_0: $\mu = 100$. We may have a decision rule that says: "Reject H_0 if the sample mean is less than 95 or more than 105."

In a courtroom we may say: "The accused is innocent until proven guilty beyond a reasonable doubt."

Errors in Hypothesis Testing

- A decision may be incorrect in two ways:
  - ✓ Type I Error: Reject a true H_0
    - The Probability of a Type I error is denoted by α.
    - α is called the **level of significance** of the test
  - ✓ Type II Error: Accept a false H_0
    - The Probability of a Type II error is denoted by β.
- α and β are conditional probabilities:
  - ✓ $\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$
  - ✓ $\beta = P(\text{Accept } H_0 \mid H_0 \text{ is false})$

Decision Table

A **contingency table** illustrates the possible outcomes of a statistical hypothesis test.

	State of Nature	
Decision	H_0 True	H_0 False
Do not Reject H_0	Correct	Type II Error (β)
Reject H_0	Type I Error (α)	Correct

The Power of a Test

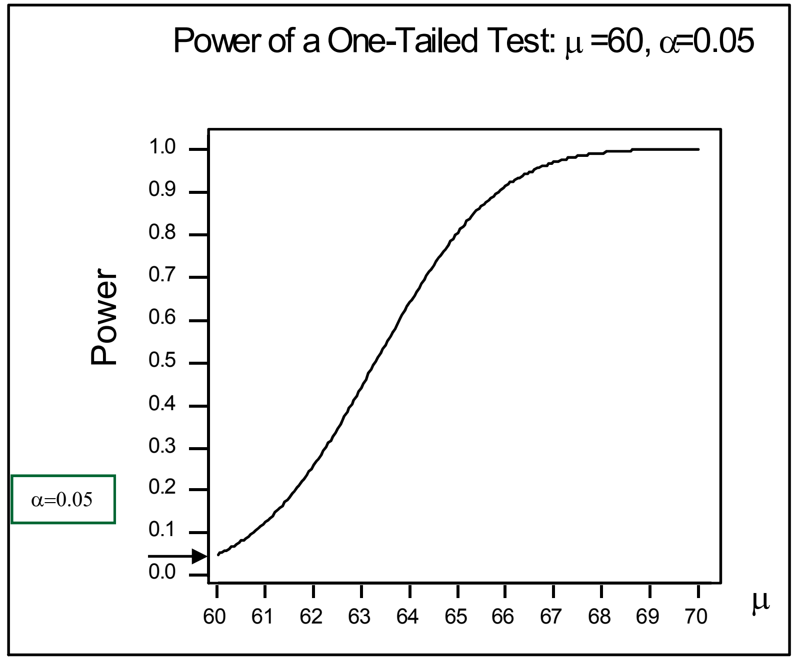
The **power** of a statistical hypothesis test is the probability of rejecting the null hypothesis when the null hypothesis is false.

$$\textbf{Power} = (1 - \beta)$$

The Power Function

The probability of a type II error, and the power of a test, depends on the actual value of the unknown population parameter. The relationship between the population mean and the power of the test is called the **power function**.

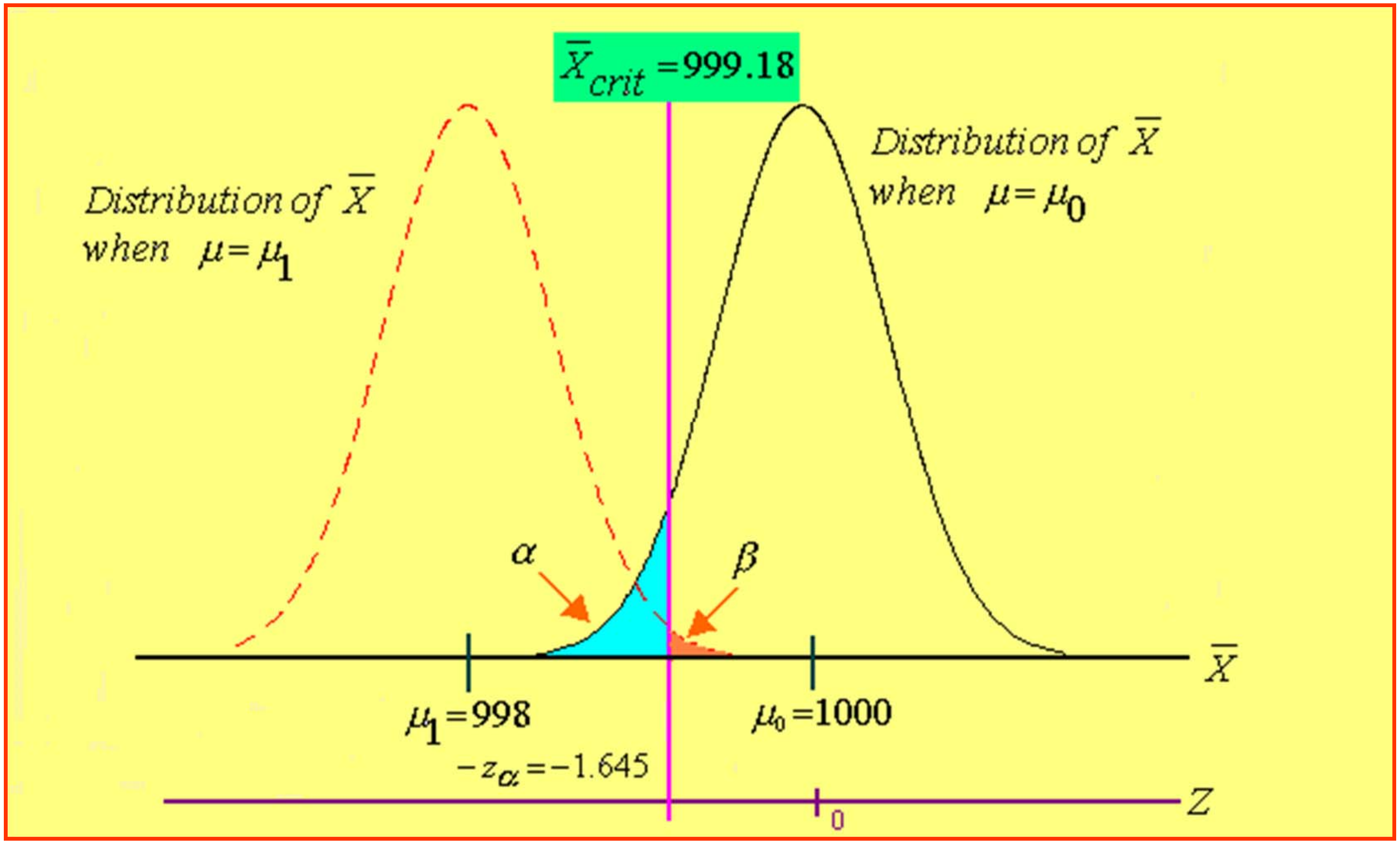
Value of μ	β	Power = $(1 - \beta)$
61	0.8739	0.1261
62	0.7405	0.2695
63	0.5577	0.4423
64	0.3613	0.6387
65	0.1963	0.8037
66	0.0877	0.9123
67	0.0318	0.9682
68	0.0092	0.9908
69	0.0021	0.9972



Factors Affecting the Power Function

- The power depends on the distance between the value of the parameter under the null hypothesis and the *true value* of the parameter in question: *the greater this distance, the greater the power*.
- *The smaller the population standard deviation, the greater the power.*
- *The larger the sample size, the greater the power*.
- *The smaller the level of significance,* α *, the smaller the power.*

Significant level α and power 1-β



Example

A company that delivers packages within a large metropolitan area claims that it takes an average of 28 minutes for a package to be delivered from your door to the destination. Suppose that you want to carry out a hypothesis test of this claim.

Set the null and alternative hypotheses:

H_0: $\mu = 28$

H_1: $\mu \neq 28$

Collect sample data:

n = 100

x = 31.5

s = 5

Construct a 95% confidence interval for the average delivery times of *all* packages:

$$\bar{x} \pm z_{.025} \frac{s}{\sqrt{n}} = 31.5 \pm 1.96 \frac{5}{\sqrt{100}}$$

$$= 31.5 \pm .98 = [30.52,\ 32.48]$$

We can be 95% sure that the average time for all packages is between 30.52 and 32.48 minutes.

Since the asserted value, 28 minutes, is not in this 95% confidence interval, we may reasonably reject the null hypothesis.

Example

An automatic bottling machine fills cola into two liter (2000 cc) bottles. A consumer advocate wants to test the null hypothesis that the average amount filled by the machine into a bottle is at least 2000 cc. A random sample of 40 bottles coming out of the machine was selected and the exact content of the selected bottles are recorded. The sample mean was 1999.6 cc. The population standard deviation is known from past experience to be 1.30 cc. Compute the *p*-value for this test.

The Concepts of Hypothesis Testing

- A **test statistic** is a sample statistic computed from sample data. The value of the test statistic is used in determining whether or not we may reject the null hypothesis.

- The **decision rule** of a statistical hypothesis test is a rule that specifies the conditions under which the null hypothesis may be rejected.

- The ***p-value*** is the probability of obtaining a value of the test statistic as extreme as, or more extreme than, the actual value obtained, when the null hypothesis is true.

Policy: When the *p*-value is less than α , reject H_0.

Testing Population Means

- Cases in which the **test statistic** is ***Z***

✓ σ is known and the population is normal.

✓ σ is known and the sample size is at least 30. (The population need not to be normal)

The formula for calculating Z is :

$$z = \frac{\bar{x} - \mu}{\left(\sigma / \sqrt{n}\right)}$$

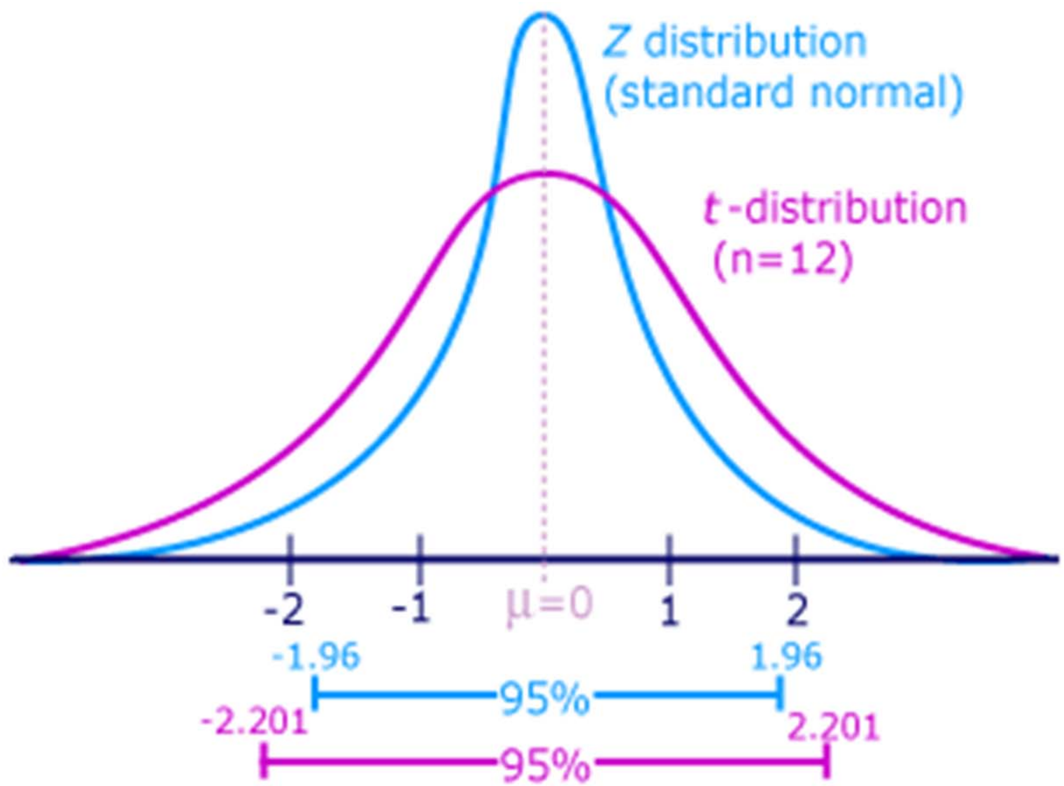
- Cases in which the **test statistic** is ***t***

✓ σ is unknown but the sample standard deviation is known and the population is normal.

The formula for calculating t is :

$$t = \frac{\bar{x} - \mu}{\left(s / \sqrt{n}\right)}$$

Normal distribution and t-distribution



Test Statistic for population mean : $\dfrac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$

If the null hypothesis μ= μ_0 is true, then $\dfrac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

A random sample of size n with sample mean $\overline{x}$ leads to an observed test statistic $z_0 = \dfrac{\overline{x}-\mu_0}{\sigma/\sqrt{n}}$

Given the significance level α:

Case 1. H_1: $\mu > \mu_0$, the rejection region is $\{z > z_\alpha\}$

p-value = $P\{Z > z_0\}$

Case 2. H_1: $\mu < \mu_0$, the rejection region is $\{z < -z_\alpha\}$

p-value = $P\{Z < z_0\}$

Case 3. H_1: $\mu \neq \mu_0$, the rejection region is $\{|z| > z_{\alpha/2}\}$

p-value = $2 \cdot P\{Z > |z_0|\}$

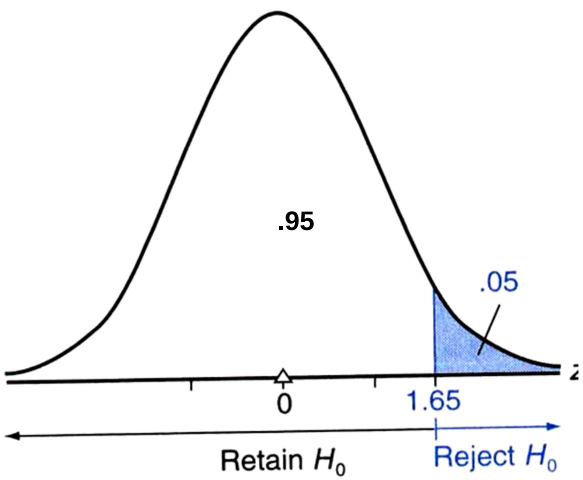
Critical Region

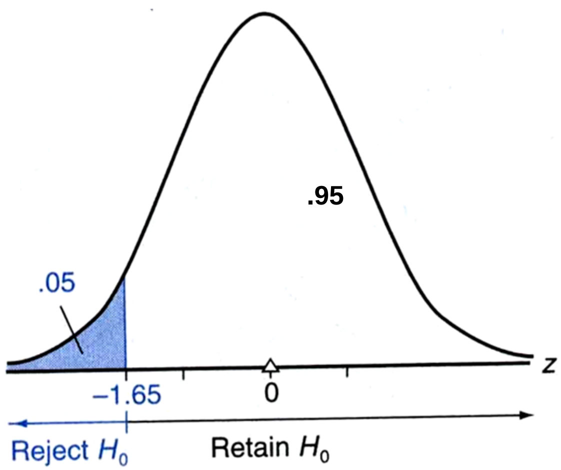
$H_0: \mu=\mu_0$ vs. $H_1: \mu>\mu_0$	$H_0: \mu=\mu_0$ vs. $H_1: \mu<\mu_0$	$H_0: \mu=\mu_0$ vs.. $H_1: \mu \neq \mu_0$

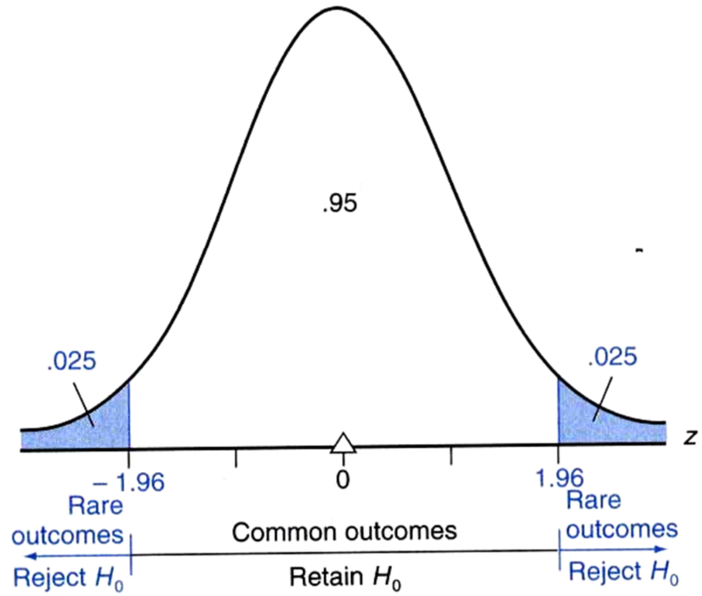
Right-tail Critical Region

Left-tail Critical Region

Two-tailed Critical Region







Test Statistic for population mean if σ is unknown : $\dfrac{\overline{X}-\mu_0}{s/\sqrt{n}}$

If the null hypothesis $\mu = \mu_0$ is true, then $\dfrac{\overline{X}-\mu_0}{s/\sqrt{n}} \sim t(n-1)$

A random sample of size n with sample mean $\overline{x}$ leads to an observed test statistic $t_0 = \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}}$

Given the significance level α:

Case 1. H_1: $\mu > \mu_0$, the rejection region is $\{t > t_\alpha(n-1)\}$

p-value = $P\{t(n-1) > t_0\}$

Case 2. H_1: $\mu < \mu_0$, the rejection region is $\{t < -t_\alpha(n-1)\}$

p-value = $P\{t(n-1) < t_0\}$

Case 3. H_1: $\mu \neq \mu_0$, the rejection region is $\{|t| > t_{\alpha/2}(n-1)\}$

p-value = $2 \cdot P\{t(n-1) > |t_0|\}$

Example

A floodlight is said to last an average of 65 hours. A competitor believes that the average life of the floodlight is less than that stated by the manufacturer and sets out to prove that the manufacturer's claim is false. A random sample of 21 floodlight elements is chosen and shows that the sample average is 62.5 hours and the sample standard deviation is 3. Using α=0.01, determine whether there is evidence to conclude that the manufacturer's claim is false.

Testing Population Proportion *p*

If the null hypothesis $p = p_0$ is true, then
$$\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} \sim N(0,1)$$

A random sample of size n with sample proportion $\hat{p}$ leads to an observed test statistic
$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

Given the significance level α:

Case 1. H_1: $p > p_0$, the rejection region is $\{z > z_\alpha\}$

p-value = $P\{Z > z_0\}$

Case 2. H_1: $p < p_0$, the rejection region is $\{z < -z_\alpha\}$

p-value = $P\{Z < z_0\}$

Case 3. H_1: $p \neq p_0$, the rejection region is $\{|z| > z_{\alpha/2}\}$

p-value = $2 \cdot P\{Z > |z_0|\}$

Example

A coin is to tested for fairness. It is tossed 200 times and only 80 Heads are observed. Test if the coin is fair at an α of 5% (significance level).

Let p denote the probability of a Head

H_0: $p = 0.5$ vs. H_1: $p \neq 0.5$ $\alpha = 5\%$

Test statistic:

Critical region:

p-value :

Confidence interval

Example

- "After looking at 1349 hotels nationwide, we've found 13 that meet our standards." This statement by the Small Luxury Hotels Association implies that the proportion of all hotels in the United States that meet the association's standards is 13/1349=0.0096.
- The management of a hotel that was denied acceptance to the association wanted to prove that the standards are not as stringent as claimed and that, in fact, the proportion of all hotels in the United States that would qualify is higher than 0.0096.
- The management hired an independent research agency, which visited a random sample of 600 hotels nationwide and found that 7 of them satisfied the exact standards set by the association.
- Is there evidence to conclude that the population proportion of all hotels in the country satisfying the standards set by the Small Luxury hotels Association is greater than 0.0096 given level $\alpha=0.10$?

H_0: ____________ vs. H_1: ____________

For $\alpha = 0.10$ the critical value

Reject H_0 if { }

The test statistic is:

Decision:

p-Value =