Section 4.6 Tests of characteristic of two distributions

1. Tests of Two Population Means

Two populations (distributions) :

$$X \sim \left(\mu_1, \sigma_1^2\right) \text{ and } Y \sim \left(\mu_2, \sigma_2^2\right)$$

Two random samples from X and Y respectively :

$$X_1, \ldots, X_{n_1} \overset{i.i.d.}{\sim} X, \text{ with sample size } n_1$$

$$Y_1, \ldots, Y_{n_2} \overset{i.i.d.}{\sim} Y, \text{ with sample size } n_2$$

Testing hypothesis :

$$H_0 : \mu_1 = \mu_2 \quad \text{vs} \quad H_1 : \mu_1 > \mu_2 \left(\mu_1 \neq \mu_2, \mu_1 < \mu_2\right)$$

Sampling Distribution of Sample Mean Difference

Case 1. Independent Normal Distributions :

$$X \sim N\left(\mu_1, \sigma_1^2\right) \text{ and } Y \sim N\left(\mu_2, \sigma_2^2\right)$$

where σ_1^2 and σ_2^2 are known.

The sampling distribution of sample mean difference is

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

Under null hypothesis : $H_0 : \mu_1 = \mu_2$, test statistic is following

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Case 1. Independent Normal Distributions (cont.)

Given significance level α, rejection region is

$$\begin{cases} \{Z > z_\alpha\} & \text{for right-tailed test } (H_1 : \mu_1 > \mu_2) \\ \{Z < -z_\alpha\} & \text{for left-tailed test } (H_1 : \mu_1 < \mu_2) \\ \{|Z| > z_{\alpha/2}\} & \text{for two-tailed test } (H_1 : \mu_1 \neq \mu_2) \end{cases}$$

Then p-value can be calculated accordingly as follows

$$P\{Z > z_o\}\ ,\ P\{Z < -z_o\},\ 2 * P\{Z > |z_o|\}$$

where z_o is the observed test statistic based on the sample.

$$[\text{ Note} : \hat{\sigma}_i^2 = s_i^2 = \frac{1}{n_i}\sum_{i=1}^{n_i}\left(X_i - \bar{X}\right)^2,\ \text{large } \mathrm{n_i} > 30, i = 1,2]$$

Case 2. Independent Normal Distributions with same variances:

$$X \sim N\left(\mu_1, \sigma_1^2\right) \text{ and } Y \sim N\left(\mu_2, \sigma_2^2\right)$$

where σ_1^2, σ_2^2 are the same, but unknown.

Let $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the pooled sample variance

$$\hat{\sigma}^2 = s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Under null hypothesis: $H_0 : \mu_1 = \mu_2$, test statistic

$$\mathrm{T} = \frac{\overline{X} - \overline{Y}}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t(n_1 + n_2 - 2)$$

Case 3. Paired sample t test (normal population):

paired data: $(X_i, Y_i), i = 1,\dots, n$

Difference $D_i = X_i - Y_i, i = 1,\dots, n$

Sample mean and sample variance:

$$\overline{D} = \frac{1}{n}\sum_{i=1}^{n} D_i, \quad s_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(D_i - \overline{D}\right)^2$$

Hypothesis: $H_0 : \mu_D = 0$ vs $H_1 : \mu_D \neq 0 (> 0, < 0)$

$$\mathrm{T} = \frac{\overline{D}}{\sqrt{\frac{s_D^2}{n}}} \sim t(n-1) \quad under\ H_0.$$

Example 4.6-3 Ten engineers' knowledge of basic statistical concepts was measured on a scale of 100 before and after a short course in statistical quality control. The engineers were selected at random. Table 4.6-1 shows the results of the tests, where $\overline{w} = 3.9$ and $s_w^2 = 31.21$; thus, the computed test statistic

$$t = \frac{3.9}{\sqrt{31.21/10}} = 2.21 \geq t(0.05; 9) = 1.833.$$

We reject $\mu_1 = \mu_2$ and accept $\mu_1 > \mu_2$ at the $\alpha = 0.05$ significance level. That is, the engineers' knowledge of basic statistical concepts seems to have increased after a course in statistical quality control.

Table 4.6-1 Statistical Knowledge Before and After Completing a Short Course in Statistical Quality Control

Engineer	Before, y	After, x	$w = x - y$
1	43	51	8
2	82	84	2
3	77	74	−3
4	39	48	9
5	51	53	2
6	66	61	−5
7	55	59	4
8	61	75	14
9	79	82	3
10	43	48	5

■

Example 4.6-3 (right tailed test) R code

```
# input data X(after) and Y(before)
  Y = c(43,82,77,39,51,66,55,61,79,43)
  X = c(51,84,74,48,53,61,59,75,82,48)

# number of observations
    n = length(y)

# paired difference – paired data
     Diff = X-Y

# sample variance of Diff
   diff.var = var(Diff)

# observed t-statistic
    tobs.diff = mean(Diff)/sqrt(diff.var/(n))
# compare with 95% t-quantile
   tobs.diff > qt(0.95, n-1)
# p-value
   pvalue.diff = 1 - pt(tobs.diff, n-1)
```

```
## assume X and Y are independent

# pooled variance
  pool.var = (var(X)+var(Y))/2

# observed t-statistic
  tobs.pool = (mean(X)-mean(Y))/sqrt(pool.var*2/n )

# compare with 95% t-quantile
    tobs.pool > qt(0.95, 2*(n-1))

# p-value
 pvalue.pool = 1- pt(tobs.pool, 2*(n-1))
```

2. Test of Two Proportions

Two independent Bernoulli distributions:

$$X \sim Bernoulli(p_1) \perp Y \sim Bernoulli(p_2)$$

Two random samples from X_1 and X_2 respectively:

$$X_{1,1}, \ldots, X_{1,n_1} \sim X_1, \text{ with sum } Y_1 = \sum_{i=1}^{n_1} X_{1,i} \sim Binomial(n_1, p_1)$$

$$X_{2,1}, \ldots, X_{2,n_2} \sim X_2, \text{ with sum } Y_2 = \sum_{i=1}^{n_2} X_{2,i} \sim Binomial(n_2, p_2)$$

Testing hypotheses:

$$H_0 : p_1 = p_2 \quad \text{vs} \quad H_1 : p_1 > p_2 (p_1 \neq p_2, p_1 < p_2)$$

Test Statistics for Proportion Difference

Under null hypothesis : $H_0 : p_1 = p_2$, the two populations X_1, X_2 have the same success rate p.

Sample proportions : $\hat{p}_i = \frac{Y_i}{n_i}, i = 1,2$

Pooled sample proportion : $\hat{p} = \frac{Y_1 + Y_2}{n_1 + n_2}$

Estimator of $(p_1 - p_2)$ is $\hat{p}_1 - \hat{p}_2$ with sampling distribution

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0,1) \quad under \quad H_0.$$

Note : Rule of thumb, $np \geq 5, n(1-p) \geq 5$.

3. Test of Two Variances

Two independent normal distributions

$$X_1 \sim N\left(\mu_1, \sigma_1^2\right) \quad \perp \quad X_2 \sim N\left(\mu_2, \sigma_2^2\right)$$

Testing hypotheses : $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$

or equivalently $H_0 : \dfrac{\sigma_1^2}{\sigma_2^2} = 1$ vs $H_1 : \dfrac{\sigma_1^2}{\sigma_2^2} \neq 1$

Sample variances : s_1^2 (of size n_1), s_2^2 (of size n_2).

$$F = \frac{s_1^2}{s_2^2} \sim F\left(n_1 - 1, n_2 - 1\right) \text{ under } \sigma_1^2 = \sigma_2^2.$$

Two-sided critical region :

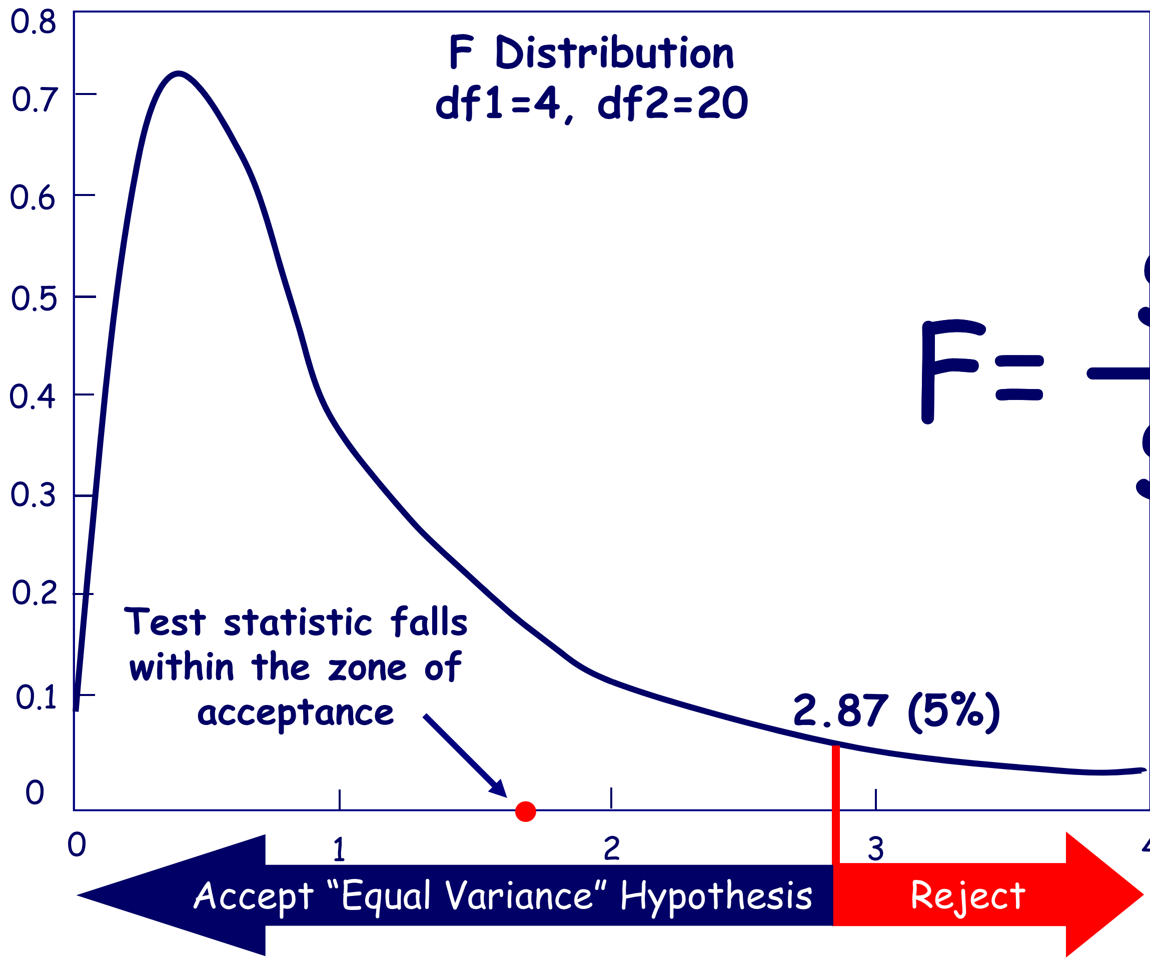
$$\left\{ F > F_{1-\frac{\alpha}{2}}\left(n_1 - 1, n_2 - 1\right), F < F_{\frac{\alpha}{2}}\left(n_1 - 1, n_2 - 1\right) \right\}$$

Sample Variance F-Test (one-tailed)

n1=5,
n2=21

s1= 5.56
s2= 4.21

F=1.744



$$F = \frac{S_1^2}{S_2^2}$$

Example 1.

- Two measurement samples with the same size 15
- Sample information:

$$\bar{x} = 11.975, s_1^2 = 36.041; \bar{y} = 8.07, s_2^2 = 22.743.$$

- Test if the two measurement means are the same.

$$H_0 : \mu_1 = \mu_2 \text{ vs } H_1 : \mu_1 \neq \mu_2$$

- If variances are the same, then used the pooled t-test; otherwise use the independent t-test

$$H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1 \text{ vs } H_0 : \frac{\sigma_1^2}{\sigma_2^2} \neq 1$$

- Test statistic $F = s_1^2 / s_2^2 = 1.58, F_{0.05}(14,14) = 2.46, F_{0.95}(14,14) = 0.4.$
- Use the pooled t-test