Section 6.3 Randomized Complete Block Design (RCBD)

Examples of RCBD

- A chemist is studying the reaction rate of five chemical agents. Only five agents can be analyzed effectively per day. Since day-to-day differences may affect the reaction rate, each day is used as a block. **All five chemical agents are tested each day in independently random orders.**
- Blocking is to arrange experimental units in groups that are similar to each other.
- Blocking factor normally is not of primary interest to the investigator, mainly to remove nuisance effects.
- Blocking factor is an observational factor, related to the characteristic of the experiemental unit, or the experimental settings. Not an experimental factor.

TABLE 7.3-1 Compressive Strength of Concrete (100 pounds per square inch)

Treatment	Batch 1	2	3	4	5	Treatment Mean
A	52	47	44	51	42	47.2
B	60	55	49	52	43	51.8
C	56	48	45	44	38	46.2
Batch mean	56	50	46	49	41	48.4

TABLE 7.3-2 Data from a Randomized Complete Block Experiment

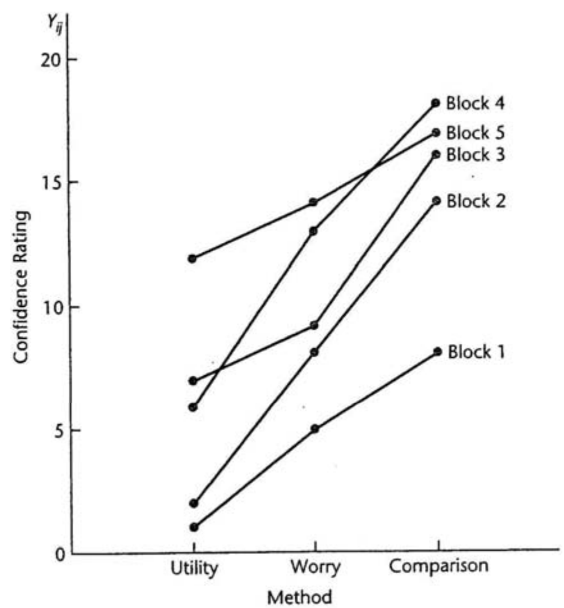
Treatment	Block 1	2	$\cdots$	j	$\cdots$	b	Treatment Mean
1	Y_{11}	Y_{12}		$\cdots$		Y_{1b}	$\bar{Y}_{1\cdot}$
2	Y_{21}	Y_{22}		$\cdots$		Y_{2b}	$\bar{Y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
i	$\cdots$	$\cdots$	$\cdots$	Y_{ij}	$\cdots$	$\cdots$	$\bar{Y}_{i\cdot}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
k	Y_{k1}	Y_{k2}		$\cdots$		Y_{kb}	$\bar{Y}_{k\cdot}$
Block mean	$\bar{Y}_{\cdot 1}$	$\bar{Y}_{\cdot 2}$	$\cdots$	$\bar{Y}_{\cdot j}$	$\cdots$	$\bar{Y}_{\cdot b}$	$\bar{Y}_{\cdot\cdot}$

In an experiment on decision making, executives were exposed to one of three methods of quantifying the maximum risk premium they would be willing to pay to avoid uncertainty in a business decision. The three methods are the utility method, the worry method, and the comparison method. After using the assigned method, the subjects were asked to state their degree of confidence in the method of quantifying the risk premium on a scale from 0 (no confidence) to 20 (highest confidence).

Fifteen subjects were used in the study. They were grouped into five blocks of three executives, according to age. Block 1 contained the three oldest executives, and so on.

	Experimental Unit		
	1	2	3
Block 1 (oldest executives)	C	W	U
2	C	U	W
3	U	W	C
4	W	U	C
5 (youngest executives)	W	C	U

C : Comparison method
W : Worry method
U : Utility method



RCBD with fixed factor and fixed block factor

- Model (with k treatments and b blocks)

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, i = 1,\ldots,k; j = 1,\ldots,b.$$

where iid $\varepsilon_{ij} \sim N(0,\sigma^2), \sum_{i=1}^{k}\tau_i = \sum_{j=1}^{b}\beta_j = 0.$

- SS Decomposition: *SSTO = SSTR + SSB+SSE*

$$SSTO = \sum_{i=1}^{k}\sum_{j=1}^{b}\left(Y_{ij} - \overline{Y}_{..}\right)^2, \overline{Y}_{..} = \frac{1}{kb}\sum_{i=1}^{k}\sum_{j=1}^{b} Y_{ij};$$

$$SSTR = \sum_{i=1}^{k} b\left(\overline{Y}_{i.} - \overline{Y}_{..}\right)^2, \overline{Y}_{i.} = \frac{1}{b}\sum_{j=1}^{b} Y_{ij}, i = 1,\ldots, k;$$

$$SSB = \sum_{j=1}^{b} k\left(\overline{Y}_{.j} - \overline{Y}_{..}\right)^2, \overline{Y}_{.j} = \frac{1}{k}\sum_{i=1}^{k} Y_{ij}, j = 1,\ldots, b;$$

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{b}\left(Y_{ij} - \overline{Y}_{i.} - \overline{Y}_{.j} + \overline{Y}_{..}\right)^2$$

ANOVA for RCBD

Source of Variation	SS	DF	MS	F
Treatment	SSTR	k-1	MSTR = SSTR/(k-1)	F_{TR} = MSTR / MSE
Block	SSB	b-1	MSB = SSB/(b-1)	F_B = MSB / MSE
Error	SSE	(k-1)(b-1)	MSE = SSE/((k-1)(b-1))	
Total	SSTO	Kb-1		

Distribution of F-statistic in ANOVA

$$E(MSE) = \sigma^2, E(MSTR) = \sigma^2 + \frac{b}{k-1}\sum_{i=1}^{k} \tau_i^{\,2},$$

$$E(MSB) = \sigma^2 + \frac{k}{b-1}\sum_{j=1}^{b} \beta_j^{\,2}.$$

By Cochran's Theorem, SSTR, SSB, and SSE are mutually independent.

Under $H_0 : \tau_i = 0, \text{i} = 1,.., \text{k}, (H_1 : \text{at least one } \tau_i \neq 0)$

$$F_{TR} = \frac{SSTR/(k-1)}{SSE/((k-1)(b-1))} \sim F(k-1,(k-1)(b-1)).$$

Under $H_0 : \beta_j = 0, \text{ j} = 1,.., \text{b}, (H_1 : \text{at least one } \beta_j \neq 0)$

$$F_B = \frac{SSB/(b-1)}{SSE/((k-1)(b-1))} \sim F(b-1,(k-1)(b-1)).$$

Parameter Estimation and Comparison Test

- ANOVA model (RCBD)

  LSE or MLE (i = 1,...k, j = 1,..., b)

  $$\hat{\mu} = \bar{Y}_{..}, \quad \hat{\tau}_i = \bar{Y}_{i.} - \bar{Y}_{..}, \quad \hat{\beta}_j = \bar{Y}_{.j} - \bar{Y}_{..}$$

  Reference Distribution :

  $$\bar{Y}_{i.} \sim N\left(\mu + \tau_i, \frac{\sigma^2}{b}\right) \quad \Rightarrow \quad \frac{\bar{Y}_{i.} - (\mu + \tau_i)}{\sqrt{\frac{MSE}{b}}} \sim t((k-1)(b-1)).$$

- 100(1-α)% Confidence Interval of *i-th* treatment mean $(\mu + \tau_i)$:

  $$\bar{Y}_{i.} \pm t_{\frac{\alpha}{2}}((k-1)(b-1)) \cdot \sqrt{\frac{MSE}{b}}$$

Follow-up Comparison and Test

- 100(1-α)% Confidence Interval of j-th block mean $(\mu + \beta_j)$

$$\bar{Y}_{.j} \pm t_{\frac{\alpha}{2}}\left((k-1)(b-1)\right) \cdot \sqrt{\frac{MSE}{k}}$$

- 100(1-α)% Confidence Interval of $(\mu_i - \mu_j)$ or $(\tau_i - \tau_j)$:

$$\left(\bar{Y}_{i.} - \bar{Y}_{j.}\right) \pm t_{\frac{\alpha}{2}}\left((k-1)(b-1)\right) \cdot \sqrt{\frac{2MSE}{b}}$$

- 100(1-α)% Tukey's Simultaneous C.I. for $(\tau_i - \tau_j)$, any pair of i,j=1,..k :

$$\left(\bar{Y}_{i.} - \bar{Y}_{j.}\right) \pm q_{\alpha}\left(k, (k-1)(b-1)\right) \cdot \sqrt{\frac{MSE}{b}}$$

Model Diagnosis

- Normality Check for iid errors -- histogram and QQ plot

$$\varepsilon_{ij} \overset{\text{iid}}{\sim} N\left(0, \sigma^2\right), \ \sum_{i=1}^{k} \tau_i = \sum_{j=1}^{b} \beta_j = 0.$$

- Variances in different treatment and block group are roughly the same

$$\bar{Y}_{i.} = \frac{1}{b} \sum_{j=1}^{b} Y_{ij} \sim N\left(\mu + \tau_i, \frac{\sigma^2}{b} \right), \quad i = 1, \ldots, k$$

$$\bar{Y}_{.j} = \frac{1}{k} \sum_{i=1}^{k} Y_{ij} \sim N\left(\mu + \beta_j, \frac{\sigma^2}{k} \right), \quad j = 1, \ldots, b$$

- Residual vs fitted value $\sum_{i=1}^{k} \sum_{j=1}^{b} e_{ij} \hat{Y}_{ij} = 0$

Example: (SAS code)

avgain is the mean average daily gain of all calves in a pen. Average daily gain is (final weight minus initial weight)/(number of days) on study.

data weightgain;
input farm pen diet $ avgain;
cards;
1 1 C 2.19
1 2 D 2.44
1 3 B 3.02
...
 	
6 21 A 1.73
6 22 D 1.49
6 23 B 1.96
6 24 C 1.33
;
	/* Without Blcok Effect*/
	proc glm data=weightgain;
	class diet;
	model avgain=diet;
	lsmeans diet / pdiff; run;
	
	/* with block effect */
	proc glm data=weightgain;
	class farm diet;
	model avgain=farm diet / p clm;
	lsmeans diet / pdiff cl adjust=tukey;
	output out=OutData p=predict r=residual;
	run;
	proc gplot data=Outdata;
	plot predict*residual avgain*residual;
	run;

Section 6.4 Design with two blocks

- Latin Square: each of k treatments is assigned to each level of each of the two blocking factors in a special way

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}.$$

- Effect model

$$Y_{i,jl} = \mu + \tau_i + \beta_j + \alpha_l + \varepsilon_{jl}, \quad i, j, l = 1,\ldots,k,$$

$$\text{where iid error } \varepsilon_{jl} \sim \mathrm{N}(0, \sigma^2), \sum_i \tau_i = \sum_j \beta_j = \sum_l \alpha_l = 0$$

Sum Squares of Treatment, Row and Column

$$Y_{i,jl} = \mu + \tau_i + \beta_j + \alpha_l + \varepsilon_{jl}, \text{ wherer } i, j, l = 1,\ldots,k$$

$$SSTO = \sum_{i=1}^{k}\sum_{j=1}^{k}\left(Y_{i,jl} - \overline{G}\right)^2, \overline{G} = \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{l=1}^{k} Y_{i,jl};$$

$$SSTR = \sum_{i=1}^{k} k\left(\overline{T_i} - \overline{G}\right)^2, \overline{T_i} = \frac{1}{k}\sum_{j=1}^{k}\sum_{l=1}^{k} Y_{i,jl}, i = 1,\ldots,k;$$

$$SSRow = \sum_{l=1}^{k} k\left(\overline{R_l} - \overline{G}\right)^2, \overline{R_l} = \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k} Y_{i,jl}, l = 1,\ldots,k;$$

$$SSCol = \sum_{j=1}^{k} k\left(\overline{C}_j - \overline{G}\right)^2, \overline{C}_j = \frac{1}{k}\sum_{i=1}^{k}\sum_{l=1}^{k} Y_{i,jl}, j = 1,\ldots,k;;$$

$$SSE = SSTO - SSTR - SSRow - SSCol$$

ANOVA for Latin Square with two blocks

Source of Variation	SS	DF	MS	F
Treatment	SSTR	k-1	MSTR	F_{TR} = MSTR / MSE
Row	SS_{Row}	k-1	MS_{Row}	$F_{Row} = MS_{Row}$ / MSE
Column	SS_{Col}	k-1	MS_{Col}	$F_{Col} = MS_{Col}$ / MSE
Error	SSE	k^2 -3k+2	MSE	
Total	SSTO	k^2 -1		

Example 4-1

The yields (in bushels per 0.1 of an acre) of 4 corn hybrids.

		Column			Row Total	Treatment Total
	10-A	14-B	7-C	8-D	39	$T_A = 53$
Row	7-D	18-A	11-B	8-C	44	$T_B = 44$
	5-C	10-D	11-A	9-B	35	$T_C = 30$
	10-B	10-C	12-D	14-A	46	$T_D = 37$
Column Total	32	52	41	39	164	

H_0: $\tau_A = \tau_B = \tau_C = \tau_D = 0$

SSTR=72.5, SSE=10.5 , F_{TR}=MSTR/MSE=13.8

k-1=3, $k^2 - 3k + 2 = 16 - 12 + 2 = 6$, $F_{0.01}(3, 6) = 9.78$, p-value<0.01.