## Homework 3 – STAT 521.

Due on Mondy, Oct. 5, 2015

**Prob. 1.** Suppose $X$ and $Y$ are both normal random variables, and

$$Y|\, X = x \sim N\left(\mu + x, \sigma^2\right), X \sim N\left(0, \lambda^2\right).$$

Find the unconditional distribution of $Y$.

**Prob. 2.** Consider linear model

$$Y = X\theta + \varepsilon, E(\varepsilon) = 0, V(\varepsilon) = \sigma^2 I_n,$$

where $X$ is a matrix of order $n \times m$ and rank $r \leq m < n$, and $\mathbf{1}_n = (1, ..., 1)' \in M(X)$. If $\hat{\theta}$ is a solution of the normal equations, show that$\mathbf{1}_n' \left(Y - X\hat{\theta}\right) = 0$.

**Prob. 3.** For a simple linear regression model with intercept only

$$Y_i = \beta_0 + \varepsilon_i, \text{where i.i.d. errors } \varepsilon_i \sim N\left(0, \sigma^2\right), i = 1, ..., n.$$

(1). Find appropriate matrix or vector for $X, \beta$ and rewrite the linear regression in the general form $Y = X\beta + \varepsilon$.

(2). Determine the orthogonal projection operator (least square) $P_X$, and calculate the predicted value $\hat{Y} = P_X Y$, and the residual $\hat{e} = Y - \hat{Y}$.

(3). Let matrix $Z$ be a matrix of order $n \times (n-1)$ such that matrix $A = \left(n^{-1/2}X, Z\right)$ is an orthogonal matrix. Find the distribution of $A'Y$ and $Z'Y$ .

(4). Based on Cochran's Theorem, prove that the sample mean $\bar{Y}$ is independent of the sample variance $s^2$. [Hint: Decompose $Y'Y$.]

**Prob. 4.** Prove the following theorem relatedto Cochran's Theorem: Let $A_i, i = 1, ..., k$ be $p \times p$ symmetric matrices such that $\sum_{i=1}^{k} A_i = I_p$.Then the following three conditions are equivalent:

(1). $A_i A_j = 0$ for $i \neq j$; (2). $A_i$ are all idempotent; (3). $\sum_{i=1}^{k} rank\left(A_i\right) = p$.

[Hint: Only need to show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.]