Homework 5 – STAT 521.

Due on Oct. 28, 2015

Prob 1. Consider normal linear model $Y \sim N_n\left(X\theta, \sigma^2 I_n\right)$, where $X_{n\times m}$ has rank $r \le m < n$. Assume σ^2 is known.

(1). Suppose $c'\theta$ is estimable. For any unbiased linear estimator $b'Y$ of $c'\theta$, find a confidence interval for $c'\theta$ centered at $b'Y$, i.e. $b'Y \pm ME(b,Y)$, where $ME(b,Y)$ is the margin of error relying on b, Y.

(2). Find the unbiased estimator $b_0'Y$ of $c'\theta$ with minimum ME.

(3). Consider the class of normed estimable functions

$$\mathcal{N} = \{c'\theta : c \in \mathcal{M}\left(X'\right), c'c = 1\}$$

Find the function $c_0'\theta$ in N which has the minimum margin of error confidence interval, i.e.

$$\min_{b:E(b'Y)=c_0'\theta} ME(b,Y) = \min_{c\in\mathcal{N}} \min_{b:E(b'Y)=c'\theta} ME(b,Y)$$

Prob 2. Consider the same normal linear model $Y \sim N_n\left(X\theta, \sigma^2 I_n\right)$, where $X_{n\times m}$ has rank $r \le m < n$. But assume σ^2 is unknown. Suppose $c'\theta$ is estimable. Let $b'Y$ be an unbiased linear estimator of $c'\theta$, not necessary BLUE. Construct a statistics $\left(b'Y - c'\theta\right)/H$ such that it follows a t distribution.

Prob 3. Consider the linear model

$$Y = X\theta + \varepsilon, E\left(\varepsilon\right) = 0, Var\left(\varepsilon\right) = \sigma^2 I_n,$$

where $X_{n\times m}$ has rank $r < m \le n$.

(1). Show that the linear zero functions $\{b \in R^n, E\left(b'Y\right) = 0\}$ forms a linear space with dimension $(n - r)$.

(2). Consider an orthonormal basis for linear space in (1), i.e. $z_1, ..., z_{n-r}$, with $z_i'z_j = \delta_{ij}$, the Kronecker delta. Show that

$$R_0^2 = \min_{\theta\in R^m} \left(Y - X\theta\right)'\left(Y - X\theta\right) = \sum_{l=1}^{n-r} \left(z_l'Y\right)^2,$$

and then construct an unbiased estimator for σ^2 as a function of $z_1'Y, ..., z_{n-r}'Y$.

(3). Suppose $Y = (Y_1, ..., Y_n)'$, $X = (X_1, ..., X_n)'$, and $E(Y_1) = E(Y_2)$. If the BLUE of an estimable function $p'\theta$ is $p'\hat{\theta} = b_1 Y_1 + b_2 Y_2 + ... + b_n Y_n$, show that $b_1 = b_2$.

Prob 4. Testing treatment effects in a general ANOVA model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, i = 1, ..., k; j = 1, ..., n,$$

where i.i.d. errors $\varepsilon_{ij} \sim N\left(0, \sigma^2\right)$. Let $\beta_1 = \mu, \beta_2 = (\alpha_1, ..., \alpha_k)'$, rewrite the model in the general matrix form as $Y = X_1\ \beta_1 + X_2\beta_2 + \varepsilon$. Find the linearly independent functions of $(I - P_{X_1})\, X_2\beta_2$ that are testable.