Homework 1 – Math 313 – Fall 2009

1. Prove the following statements (only using the axioms for the real numbers given in the lecture and using the statements proved there). At each step say which axiom you use.

   (a) For any $x, y \in \mathbb{R}$ the equality $(-x) \cdot y = -(x \cdot y)$ holds.

   (b) For any $x, y \in \mathbb{R}$ with $x < y$ there exists $z \in \mathbb{R}$ with $x < z < y$.

   (c) For any $x > 0$ and $0 \leq h < 1$ we have $(x + h)^2 \leq x^2 + h(2x + 1)$.

   (d) For any $x > 0$ and $p > 0$ with $x^2 < p$ there exists $y > x$ with $y^2 < p$.

2. Using the notes from the lecture, write a complete proof of the following statement: For every $p \geq 0$ there exists a unique $x \geq 0$ with $x^2 = p$. (Such x is commonly denoted $\sqrt{p}$.)

3. Prove that $\sqrt{5}$ is irrational, i.e. there do not exist $m, n \in \mathbb{Z}$ with $n \neq 0$ and such that $\sqrt{5} = \frac{m}{n}$.

4. Prove that for every natural number the inequality $2n < 3^n$ holds.

5. Define a sequence $b_1, b_2, b_3, \ldots$ as follows: $b_1 = 1$, and $b_2 = 2$. If b_{n-1} and b_n are defined, let $b_{n+1} = 2b_n - b_{n-1}$. Find a formula for b_n, and prove that your formula is correct.

Due date: Friday, September 4, 2009, at noon, **no late homework will be accepted.**