Practice Questions – Math 313 – Fall 09

1. Let (a_n) be the sequence given by

$$a_n := \left(1 + \frac{1}{n^2}\right)^n.$$

   Does (a_n) converge? If yes, what is its limit?

2. Show that the series

$$\sum_{n=1}^{\infty} \frac{2n+3}{(n+1)(n+2)}$$

   does not converge.

3. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series and $\sum_{n=1}^{\infty} b_n$ be a convergent series (not necessarily absolutely convergent). Does the series

$$\sum_{n=1}^{\infty} a_n b_n$$

   converge absolutely? You must justify your answer.

4. Does the equation

$$\exp(x-2) = x^2$$

   have a solution on the interval $[0, 4]$? You must justify your answer.

5. Determine whether the following limits exist. Find the limit if it does exist:

   (a) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

   (b) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

6. Let $f : [0, 1] \to \mathbb{R}$ be the function given by

$$f(x) := \begin{cases} 1 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } x \neq \frac{1}{2^n} \text{ for any } n \in \mathbb{N}. \end{cases}$$

   (a) Let P be the partition of $[0, 1]$ given by $P := \{0, \frac{1}{8}, \frac{5}{8}, 1\}$. Compute the lower sum and the upper sum of f for this partition.

   (b) Find a partition P of $[0, 1]$ such that

$$U(f, P) - L(f, P) \le \frac{1}{4}.$$

7. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define a function $F : [0, \infty) \to \mathbb{R}$ by

$$F(x) := \int_0^x f.$$

Prove that F has a right-derivative at every $x_0 \geq 0$, that is, prove that the limit

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0}$$

exists.

*(b) Does the right-derivative exist at every $x_0 \geq 0$ if f is merely Riemann integrable over $[-a, a]$ for all $a > 0$?

8. Let $f : [-1, 1] \to \mathbb{R}$ be the function defined by

$$f(x) := \begin{cases} 0 & : \quad x \leq 0 \\ \frac{1}{q} & : \quad x = \frac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ and } p \text{ and } q \text{ have greatest common divisor } 1 \\ 0 & : \quad x \text{ irrational.} \end{cases}$$

Is f differentiable at the point $x_0 = 0$? You must justify your answer.

9. Let $f : (-1, 1) \backslash \{0\} \to \mathbb{R}$ be the function given by

$$f(x) := \begin{cases} \frac{e^x - 1}{x} & : \quad x \in (-1, 0) \\ \frac{3(x+1)}{x^2 - 4x + 3} & : \quad x \in (0, 1) \end{cases}$$

(a) Show that $\lim_{x \to 0} f(x)$ exists.

(b) Extend f to a function $\bar{f} : (-1, 1) \to \mathbb{R}$ in such a way that $\bar{f}$ is continuous. That is, define a function $\bar{f} : (-1, 1) \to \mathbb{R}$ in such a way that $\bar{f}(x) = f(x)$ for all $x \in (-1, 1) \backslash \{0\}$ and such that $\bar{f}$ is continuous. Prove that your $\bar{f}$ is continuous on $(-1, 1)$.

Note: The above problems as well as other questions will be discussed during the problem solving session on Monday, December 7, 4 – 5pm, in SEO 512.