Review Part I – Math 313 – Fall 09

The real numbers

- Axioms for the real numbers $\mathbb{R}$
- $\sup A$ and $\inf A$ for subsets $A \subset \mathbb{R}$

Sequences of real numbers

- Definition of convergence to a real number
- Definition of convergence to ∞ or $-\infty$
- If $a_n \to a$ then $|a_n| \to |a|$; the converse is not true, but $|a_n| \to 0$ implies $a_n \to 0$
- If (a_n) converges to a real number then (a_n) is bounded; the converse is not true
- Sum, difference, product, and quotients of sequences
- If $a_n \le b_n$ for all n and (a_n) and (b_n) converge then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$
- If $a_n \to \pm\infty$ then $\frac{1}{a_n} \to 0$; the converse is not true
- Sandwich theorem
- If (a_n) is non-decreasing and bounded from above then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$; analogous statement if (a_n) is non-increasing and bounded from below
- If (a_n) is non-decreasing and not bounded from above then $a_n \to \infty$; analogous statement if (a_n) is non-increasing and not bounded from below
- (a_n) converges to a real number if and only if (a_n) is a Cauchy sequence
- If (a_n) is bounded then there exists a subsequence of (a_n) which converges; the converse is not true
- Definition of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ and properties
- Principal examples of sequences:
  - $\frac{1}{n} \to 0$
  - (b^n) converges to a real number if and only if $b \in (-1, 1]$; if $|b| < 1$ then $b^n \to 0$
  - If $b > 1$ then $b^n \to \infty$
  - $a_n := \left(1 + \frac{1}{n}\right)^n \to e$, $e = 2.7182\ldots$; in class we proved that (a_n) is increasing and bounded above by 3 (recall the proof)
  - If $b > 0$ then $b^{\frac{1}{n}} \to 1$
  - $n^{\frac{1}{n}} \to 1$
  - For $x > 0$ and $a_0 > 0$ and a_n is defined by $a_n := \frac{1}{2}(a_{n-1} + \frac{x}{a_{n-1}})$ then $a_n \to \sqrt{x}$ (recall the proof)

Some important identities and inequalities

- $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- $(1+x)^n \geq 1 + nx$ whenever $x \geq -1$

Series

- Definition of convergence of $\sum_{n=1}^{\infty} a_n$ to a real number or to ∞
- Definition of absolute convergence and conditional convergence
- If $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} a_n$ converges; the converse is not true
- If $\sum_{n=1}^{\infty} a_n$ converges to a real number then $a_n \to 0$; the converse is not true
- $\sum_{n=1}^{\infty} a_n$ converges to a real number if and only if $\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}$ such that

$$\left| \sum_{n=M}^{N} a_n \right| \leq \varepsilon \quad \text{for all } N \geq M \geq N_0$$

- If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} a_n$ converges to a real number if and only if the sequence (S_N) given by $S_N := \sum_{n=1}^{N} a_n$ is bounded
- Comparison theorem
- Ratio test
- If (a_n) is non-increasing with $a_n \geq 0$ for all n and $a_n \to 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges
- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for every $x \in \mathbb{R}$; we define $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- Whenever $|x| \leq 1 + \frac{N}{2}$ then

$$\left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \leq 2 \frac{|x|^{N+1}}{(N+1)!}$$

- Principal examples of series:
  - For $p \in \mathbb{R} \backslash \{1\}$ we have $\sum_{n=0}^{N} p^n = \frac{1-p^{N+1}}{1-p}$
  - $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ (recall the proof)
  - $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (recall the proof)
  - $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges conditionally
  - For all $k \in \mathbb{N}$ and $|a| < 1$ the series $\sum_{n=1}^{\infty} n^k a^n$ converges absolutely
  - See homework assignments for more examples and techniques

Limits of functions

- Definition (using ε, δ) of $\lim_{x \to x_0} f(x)$ for a function $f : D \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $D \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset$ for all $\delta > 0$

  Note that f need not be defined at x_0, if defined then $f(x_0)$ is of no importance for existence/value of $\lim_{x \to x_0} f(x)$

- Definitions of

$$\lim_{x \to x_0^-} f(x), \quad \lim_{x \to x_0^+} f(x), \quad \lim_{x \to x_0} f(x) = \infty, \quad \lim_{x \to \infty} f(x), \quad \text{etc.}$$

- $\exists \lim_{x \to x_0} f(x)$ if and only if $\exists \lim_{x \to x_0^-} f(x), \lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$
- If $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing and bounded from above then $\exists \lim_{x \to \infty} f(x)$
- Limits of sums, differences, products, quotients of functions