Solutions ~ Midterm 2 - Math 313

① Let $\varepsilon > 0$. Set $\delta := \min\{\frac{\varepsilon}{3}, 1\} \Rightarrow \delta > 0$

For $x \in \mathbb{R}$ with $0 < |x-1| \leq \delta$ it follows that

- $|x+1| \leq 3$ and $x \neq 1$

and thus

$$|f(x)-1| = |x^2-1| = |x-1|\cdot|x+1| \leq 3\cdot\delta = \varepsilon.$$

② (a) We have for each $n \geq 1$

$$0 < \frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot \ldots \cdot 2\cdot 1}{n\cdot n \cdot \ldots \cdot n\cdot n} \leq \frac{2}{n^2}. \quad (*)$$

For $N \geq 1$ set

$$S_N := \sum_{n=1}^{N} \frac{n!}{n^n}$$

$$S_N' := 2\sum_{n=1}^{N} \frac{1}{n^2}$$

We proved in class that S_N' is bdd.

Since $S_N \le 2 S_N'$ by $(*)$ it follows that S_N is bdd.

Since $\frac{n!}{n^n} \ge 0 \ \forall n$ and thus S_N is non-decreasing it follows that S_N converges.

(b) We have for every $n \ge 1$

$$0 < \sqrt{n^2 + 1/n} - n = \frac{\left(\sqrt{n^2+1/n} - n\right)\cdot\left(\sqrt{n^2+1/n}+n\right)}{\sqrt{n^2+1/n}+n}$$

$$= \frac{n^2 + 1/n - n^2}{\sqrt{n^2+1/n}+n}$$

$$= \frac{1}{n^2 + n\cdot\sqrt{n^2+1/n}}$$

$$\le \frac{1}{n^2} \qquad (**)$$

Set.

$$S_N := \sum_{n=1}^{N} \sqrt{n^2+\tfrac{1}{n}} - n$$

$$S_{N^*}' := \sum_{n=1}^{N} \frac{1}{n^2}.$$

We know that (S_N') is bdd.

By (**), (S_N) is non-decreasing and

$$S_N \leq S_N',$$

~~Therefore~~ in particular, (S_N) is bdd.

$\Rightarrow$ (S_N) converges.

3 Since $\sum_{n=1}^{\infty} a_n$ converges it follows that

$$a_n \longrightarrow 0 \quad \text{as } n \to \infty.$$

It follows $\exists\, n_0$ s.th. $|a_n| \leq 1 \quad \forall n \geq n_0$

and in particular

$$(***) \quad a_n^4 = |a_n|^4 \leq |a_n| \qquad \forall n \geq n_0.$$

Let $K > 0$. ~~Since~~ $\cancel{\sum |a_n|^4 = \infty}$

Set $K' := K + |a_1|^4 + \dots + |a_{n_0}|^4$

Since $\sum_{n=1}^{\infty} |a_n|^4 = \infty$ it follows that

there exists N_0 s.th.

$$\sum_{n=1}^{N} |a_n|^4 \geq K^{1} \quad \forall N \geq N_0$$

Clearly $N_0 \geq n_0+1$ and thus

$$\sum_{n=n_0+1}^{N} |a_n|^4 \geq K$$

It follows from (***) that

$$\sum_{n=1}^{N} |a_n| = |a_1| + \_ + |a_{n_0}| + \sum_{n=n_0+1}^{N} |a_n|$$

$$\geq \sum_{n=n_0+1}^{N} |a_n|^4$$

$$\geq K$$

for all $N \geq N_0$.

Therefore $\sum_{n=1}^{\infty} |a_n| = \infty$.

4 Let $x_0 \in D$ and let $\varepsilon > 0$.

Set $\delta := \frac{\varepsilon}{L}$.

Then for all $x \in D$ with $|x - x_0| \leq \delta$ we have

$$|f(x) - f(x_0)| \leq L \cdot |x - x_0| \leq L \cdot \delta = \varepsilon.$$