Take Home Final Exam – Math 446 – Spring 2009

Please read the following instructions carefully:

- This is a take home exam. You should complete the exam by yourself with no help or discussions with anybody else. You may use the notes from class, the book "Topology, 2nd Edition, by J. Munkres", and the handouts distributed in the second-last week, **but no other sources**.
- You must justify your answers. Answers and results without justification will not be given credit. When giving justifications you may use all results we covered in class and you should say which facts you use.
- You may ask me questions only about notation. Send an email. I will not have office hours during the exam period.
- You must return your solutions until **Friday, May 8th, 2009**. I will not be at UIC, therefore you should put your solutions into an envelope and slide it under my door. **Please make a photocopy of your exam before handing it in, in case it gets lost.**

1. (15 points) Let X be a topological space and $x_0 \in X$. Given $[f_0], [f_1] \in \pi_1(X, x_0)$, define continuous maps $f'_j : S^1 \to X$, $j = 0, 1$, in such a way that $f'_j(e^{2\pi i s}) = f_j(s)$ for all $s \in [0, 1]$. Prove that f'_0 and f'_1 are homotopic if and only if $[f_0]$ and $[f_1]$ are conjugate in $\pi_1(X, x_0)$.

2. Let X be a topological space and A a subspace of X. Denote by $\iota : A \to X$ the inclusion map, and let $\varrho : X \to A$ be a continuous map. Suppose there exists a homotopy $H : X \times [0, 1] \to X$ between $\iota \circ \varrho$ and the identity map on X.

   (a) (5 points) Show that if ϱ is a retraction, then ι_* is an isomorphism;

   (b) (10 points) Show that if $H(A \times [0, 1]) \subset A$ then ι_* is an isomorphism;

   (c) (5 points) Find an example such that ι_* is not an isomorphism.

3. Recall that a group G is said to be finitely generated if there exist finitely many elements $g_1, \dots, g_n \in G$ such that every element in G is a finite product of the $g_1, \dots, g_n$.

   (a) (20 points) Let X be a compact metric space and suppose that X is path-connected, locally path-connected, and semi-locally simply connected. Show that $\pi_1(X)$ is finitely generated.

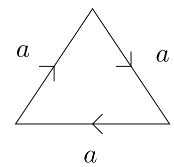
   (b) (5 points) Find an example of a compact, path-connected topological space whose fundamental group is not finitely generated.

4. (15 points) Let X and Y be the following subsets of $\mathbb{R}^3$:

$$X := \{(s, t, 0) : s^2 + t^2 = 1\} \quad \text{and} \quad Y := \{(s, 0, t) : (1 - s)^2 + t^2 = 1\}.$$

Compute the fundamental group of the subspace $Z := \mathbb{R}^3 \backslash (X \cup Y)$.

5. (30 points) Let X be the quotient space obtained from a triangle by identifying all its sides along the arrows indicated in the picture below.



Let $x_0 \in X$ be a point in the interior of the triangle. Let furthermore y_0 be an arbitrary point in $\mathbb{R}\mathrm{P}^2$ and define Y to be the space obtained from gluing X to $\mathbb{R}\mathrm{P}^2$ at the points x_0 and y_0. Find all connected covering spaces of Y.

6. Let X be a topological space, A a nonempty subspace of X, and denote by $\iota : A \to X$ the inclusion map.

(a) (5 points) Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X;

(b) (10 points) Show that $H_1(X, A) = 0$ if and only if $\iota_* : H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.