Solution to problem 8 – Math 446 – Spring 09

Proposition 1. *Let $G := \mathbb{Z}_2 * \mathbb{Z}_2$ and denote by a and b, respectively, the generators of each $\mathbb{Z}_2$. Then up to conjugacy every non-trivial subgroup of G is of one of the following forms:*

(1) $\langle a\rangle$ *or* $\langle b\rangle$*;*
(2) $\langle (ab)^n\rangle$ *for some* $n \geq 1$*;*
(3) $\langle a, (ba)^m b\rangle$ *or* $\langle b, (ab)^m a\rangle$ *for some* $m \geq 0$.

Proof. For a given non-trivial subgroup H of G denote by $l(H)$ the length of a shortest non-trivial element in H. We distinguish three cases: First, assume that $l(H)$ is even and let $w \in H$ be a shortest non-trivial element. Then, possibly after passing to a conjugate subgroup, we may assume that $w = (ab)^n$ with $2n = l(H)$. It follows that $\langle (ab)^n\rangle \subset H$. Now, let $w' \in H$ be another non-trivial element different from w. Since w' has at least length $2n$, it is either of the form $w' = (ab)^{mn+k}$ or $w' = (ba)^{mn+k}$ or $w' = (ab)^{mn+k}a$ or $w' = (ba)^{mn+k}a$ for suitable $m \geq 1$ and $0 \leq k < n$. However, after multiplying w' by w^m or w^{-m} on the right or left yields a non-trivial element of H whose length is either $2k$ or $2k + 1$. Since w is the shortest non-trivial element we conclude that $k = 0$ and $w'(ab)^{mn}$ or $w' = (ba)^{mn}$. This proves that $H = \langle (ab)^n\rangle$ and thus that H is conjugate to the subgroup in (ii). Next, suppose that $l(H)$ is odd. Let w be a non-trivial element of shortest length. Then w is of the form $w = (ab)^n a$ or $w = (ba)^n b$ for some $n \geq 0$. It is clear that after passing to a conjugate subgroup we may assume that $w = a$ or $w = b$. We may assume without loss of generality that $w = a$. Now, if H is cyclic, then it is conjugate to one of the subgroups in (i). On the other hand, if H is not cyclic then there exists a non-trivial element w' of H of shortest length with $w \neq a^m$. Clearly, w' is of the form $w' = (ba)^k b$ for some $k \geq 0$ and hence $\langle a, (ba)^k b\rangle \subset H$. Let w'' be an arbitrary element of H. We wish to show that w'' can be written as a product of a's and $(ba)^k b$'s. After possibly multiplying on the left and/or right by a we may assume that w'' is of the form $w'' = (ba)^n b$ for some $n \geq 0$. Clearly, we must have $n \geq k$. If $n > k$ then we can rewrite w'' as $w'' = (ba)^k b[a(ba)^{n-k-1}b]$. After multiplying on the left by $(ba)^k b$ we obtain the word $a(ba)^{n-k-1}b$. We can apply the above procedure again and again until we obtain the trivial element. Thus w'' can indeed be written as the product of a's and $(ba)^k b$'s, which proves that $H = \langle a, (ba)^k b\rangle$ and shows that H is conjugate to one of the subgroups in (iii). This concludes the proof. □