Final exam – Math 549 – Fall 2009

(1) Let (M, g) be a Riemannian manifold and ω a 1-form on M. Show that there exists a unique smooth vector field $Y_\omega \in \Gamma(TM)$ such that

$$\omega(x)(v) = g(x)(Y_\omega(x), v)$$

for all $x \in M$ and all $v \in T_x M$.

(2) Let $\mathbb{R}^2$ be endowed with the standard inner product $< \cdot, \cdot >$. Give an example of an affine connection $D_X Y$ on the Riemannian manifold $(\mathbb{R}^2, < \cdot, \cdot >)$ which is metric but not torsion-free.

(3) Let M be a smooth manifold and let $L : \Gamma(TM) \to \Gamma(TM)$ be a $C^\infty(M)$-linear map. Fix $x \in M$. Without using the "tensoriality theorem" prove that if X, Y are vector fields satisfying $X(x) = Y(x)$ then

$$L(X)(x) = L(Y)(x).$$

(4) Let ω be the 1-form on $\mathbb{R}^2 \backslash \{(0, 0)\}$ given by

$$\omega(x, y) = \frac{1}{x^2 + y^2} (xdy - ydx).$$

(a) Prove that ω is closed.

(b) Prove that ω is not exact.

(5) Let $M, N \subset \mathbb{R}^{k+1}$ be compact, oriented submanifolds without boundary, of dimension m and n, respectively. Suppose M and N are disjoint as subsets. Then

$$\lambda : M \times N \to S^k, \quad (x, y) \mapsto \frac{x - y}{\|x - y\|}$$

is a well-defined smooth map between oriented manifolds. If $m + n = k$ then we define the linking number of M and N by

$$\mathrm{lk}(M, N) := \deg(\lambda),$$

where $\deg(\lambda)$ is the Brouwer degree of λ.

(a) Prove that

$$\mathrm{lk}(N, M) = (-1)^{(m+1)(n+1)} \,\mathrm{lk}(M, N).$$

(b) Assume that M is the boundary of a compact oriented submanifold $X \subset \mathbb{R}^{k+1}$. Prove that if $X \cap N = \emptyset$ then $\mathrm{lk}(M, N) = 0$.

(c) Let

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\},$$
$$N := \{(x, y, z) \in \mathbb{R}^3 : (x - 1)^2 + z^2 = 1, y = 0\}.$$

Choose orientations of M and N and compute $\mathrm{lk}(M, N)$.

Due date: Wednesday, December 9, at 3pm, in my mailbox.