Homework 7 – Math 549 – Fall 2009

(1) Let M and N be oriented manifolds and assume that M has no boundary.

   (a) Prove that $M \times N$ is orientable and carries an orientation induced by the orientations on M and N.

   (b) Compare the orientations of $\partial(M \times [0,1]) = (M \times \{0\}) \cup (M \times \{1\})$ with the orientation of M.

(2) (a) Prove that the composition of two orientation preserving diffeomorphisms is orientation preserving.

   (b) Prove that an orientable, connected manifold has precisely two orientations.

   (c) Give an example of a manifold with more than two orientations.

(3) Let M be an orientable manifold without boundary, and let $f : M \to \mathbb{R}$ be a smooth function with 0 as a regular value.

   (a) Prove that $\{f \geq 0\}$ is an orientable manifold.

   (b) Prove that $\{f = 0\}$ is an orientable manifold.

(4) In the proof that for m even, there is no nowhere vanishing vectorfield on S^m, we used the following fact, which you should prove in this exercise.

   Problem: Suppose X is a vectorfield on S^m with $\|X(x)\| = 1$ for all $x \in S^m$. Given $\varepsilon > 0$, set $r := \sqrt{1+\varepsilon^2}$, and denote by S^m_r the sphere or radius r in $\mathbb{R}^{m+1}$. Define a smooth map $f_\varepsilon : S^m \to S^m_r$ by $f_\varepsilon(x) := x + \varepsilon X(x)$. Prove that for all $\varepsilon > 0$ small enough, the map f_ε is a diffeomorphism.

Due date: Will be discussed during problem session, Wednesday, December 2nd.