Representation Theory Notes

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Spring 2009

Contents

Abstract

These notes serve as an external (albeit loose) point of reference for Math 518, offered in Spring 2009. The author follows and summarizes several sections of two texts used in the course. The first section follows Warner's *Foundations of Differentiable Manifolds and Lie Groups*, while the last two sections are from Baker's *Matrix Groups: An Introduction to Lie Group Theory*. These notes are loosely organized and do not always follow the linear path of the two texts. Rather, here is presented material that the author found particular interesting or needed further clarification on at times.

1 Manifolds

We begin by introducing some basic definitions, terminology, and notation in our discussion. Differentiable manifolds can be seen as locally similar to Euclidean space and are rich enough in structure so that familiar and fundamental concepts from calculus translate over.

1.1 Differentiable Manifolds

Definition 1. Suppose $U \subset \mathbb{R}^d$ is open, and let $f : U \to \mathbb{R}$. We say that f is **differentiable of class C^k on U** (also may say that f is C^k), for k a non-negative integer, if the partial derivatives $\frac{\delta^\alpha f}{\delta r^\alpha}$ exists and are continuous on U for $[\alpha] \leq k$. For example, f is C^0 if f is continuous. Now, if $f : U \to \mathbb{R}^n$, then f is **differentiable of class C^k** if each of the component functions $f_i = r_i \circ f$ is C^k. We say that f is C^∞ if it is C^k for all $k \geq 0$.

Definition 2. A locally Euclidean space M of dimension d is a Hausdorff topological space M for which each point has a neighborhood homeomorphic to an open subset of the Euclidean space $\mathbb{R}^d$. If ϕ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^d$, ϕ is called a **coordinate map**, and the functions $x_i = r_i \circ \phi$ are called the **coordinate functions**. Also, the pair (U, ϕ) is called a **coordinate system**. Note that (U, ϕ) is also at times denoted by $(U, x_1, \ldots, x_d)$. A coordinate system (U, ϕ) is called a **cubic coordinate system** if $\phi(U)$ is an open cube about the origin in $\mathbb{R}^d$. If $m \in U$ and $\phi(m) = 0$, then the coordinate system is said to be **centered at** m.

Definition 3. A **differentiable structure $\mathcal{F}$ of class C^k** $(1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ satisfying the following properties:

1. $\bigcup_{\alpha \in A} U_\alpha = M$.
2. $\phi_\alpha \circ \phi_\beta^{-1}$ is C^k for all $\alpha, \beta \in A$.
3. The collection $\mathcal{F}$ is maximal with respect to the above. That is, if (U, ϕ) is a coordinate system such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \phi) \in \mathcal{F}$.

Note that if $\mathcal{F}_0 = \{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ is a collection of coordinate systems satisfying the first two items above, then there is a *unique* differentiable structure $\mathcal{F}$ that contains $\mathcal{F}_0$. To see this in a more explicit fashion, let

$$\mathcal{F} = \{(U, \phi) | \phi \circ \phi_\alpha^{-1} \text{ and } \phi_\alpha \circ \phi^{-1} \text{ are } C^k \text{ for all } \phi_\alpha \in \mathcal{F}_0\}.$$

Then we see that $\mathcal{F}$ contains $\mathcal{F}_0$ and satisfies the first two properties as in the above. As $\mathcal{F}$ is maximal by construction, it is a differential structure containing $\mathcal{F}_0$. Then it is clear that $\mathcal{F}$ is unique in this regard.

Let us quickly note two other basic types of differential structures on locally Euclidean spaces, the structure of the class C^* and the complex analytic structure. For a **differential structure of class** C^*, we need that the compositions given by the send item above are local given by convergent power series. For a **complex analytic structure** on a 2-dimensional locally Euclidean space, we require that coordinate systems have range in the space $\mathbb{C}^d$ and overlap holomorphically.

A d**-dimensional differentiable manifold of class** C^k is a pair $(M, \mathcal{F})$ consisting of a d-dimensional, second-countable, locally Euclidean space M together with a differentiable structure $\mathcal{F}$ of class C^k. Typically we denote this pair simply by M, under the premise that we are considering the locally Euclidean space M with some differentiable structure $\mathcal{F}$ when we speak of the differentiable manifold M. In this discussion, we restrict ourselves to the class C^∞. The term **smooth** also applies to the differentiability of C^∞. We often shorten our term to simply *manifolds* when speaking of differentiable manifolds, and we assume differentiability of class C^∞. Thus, a manifold can be seen as consisting of three objects: an underlying point set, a second-countable, locally Euclidean topology on the point set, and a differentiable structure. If X is a set, we define a **manifold structure on** X to be a choice of both a second-countable, locally Euclidean topology on X and a differentiable structure.

In the below, M and N denote differentiable manifolds, and M^d indicates that the manifold M has dimension d. Let us consider some examples.

1.2 Examples

Example 1. We obtain the standard differentiable structure on Euclidean space $\mathbb{R}^d$ by taking $\mathcal{F}$ to be the maximal collection containing $(\mathbb{R}^d, i)$, where $i : \mathbb{R}^d \to \mathbb{R}^d$ is the identity map.

Example 2. If V is a real vector space of finite dimension, then V has a natural manifold structure. Let $\{e_i\}$ be a basis for V. Then the elements of the dual basis are coordinate functions on a global coordinate system on V. Such a system uniquely determines a differentiable structure $\mathcal{F}$ on V, and this structure is independent of the choice of basis, as different bases give C^∞ overlapping coordinate systems. Interestingly enough, it turns out that the change of coordinates is given by a constant, non-singular matrix.

Example 3. If U is an open subset of a differentiable manifold $(M, \mathcal{F}_M)$, then U is itself a differentiable manifold. The differentiable structure for U is then given by

$$\mathcal{F}_U = \{(U_\alpha \cap U, \phi_\alpha | U_\alpha \cap U) | (U_\alpha, \phi_\alpha) \in \mathcal{F}_M\}.$$

1.3 Differentiable Maps

Definition 4. Let $U \subset M$ be open. We say that $f : U \to \mathbb{R}$ is a C^∞ **function on** U (denoted by $f \in C^\infty(U)$) if $f \circ \phi^{-1}$ is C^∞ for each coordinate map ϕ on M. A continuous map $\xi : M \to N$ is said to be **differentiable of class** C^∞ ($\xi \in C^\infty(M, N)$) if $g \circ \xi$ is a C^∞ function on ξ^{-1} (domain of g) for all C^∞ functions g defined on open sets in N. Equivalently, the continuous map ξ is C^∞ if and only if $\phi \circ \xi \circ \rho^{-1}$ is C^∞ for each coordinate map ρ on M and ϕ on N.

Note that the composition of two differentiable maps is differentiable, and observe that a map $\phi : M \to N$ is C^∞ if and only if for each $m \in M$ there exists an open neighborhood $U of m$ such that ϕ restricted to U is C^∞.

1.4 Tangent Vectors and Differentials

One may view a vector v with components $v_1, \ldots, v_d$ at a point p in Euclidean space $\mathbb{R}^d$ as an operator on differentiable functions. In specific, if f is differentiable on a neighborhood of p, then v assigns to f the number $v(f) \in \mathbb{R}$. We see that $v(f)$ is the directional derivative of f pointing in the direction of v at p. The operator v satisfies the following properties:

$$v(f + \lambda g) = v(f) + \lambda v(g),$$

$$v(fg) = f(p)v(g) + g(p)v(f),$$

whenever f and g are differentiable near p ($\lambda \in \mathbb{R}$). The first item says that v acts linearly on functions, while the second declares v a derivation.

Definition 5. Let $m \in M$. Functions f and g on open sets containing m are said to have the same **germ** at m if they agree on some neighborhood of m. This induces an equivalence relation on the C^∞ functions defined on neighborhoods of m. That is, two functions are considered equivalent if and only if they have the same germ. We call the equivalence classes **germs**, and we denote the set of germs at m by $\overline{F}_m$. if f is a C^∞ function on a neighborhood of m, then $\mathbf{f}$ will denote its germ. Note that the operations of addition, scalar multiplication, and the multiplication of functions induce an algebraic structure over $\mathbb{R}$. A germ $\mathbf{f}$ has a well-defined value $\mathbf{f}(m)$ at m. This value is the value at m of any representation of the germ. Suppose $F_, \subset \overline{F}_m$ is the set of germs that vanish at m. Then we see that F_m is an ideal, and denote by F_m^k its kth power. F_m^k is the ideal of $\overline{F}_m$ that is comprised of all finite linear combinations of k-fold products of elements of F_m, which form an ascending sequence of ideals

$$\cdots \subset F_m^3 \subset F_m^2 \subset F_m \subset \overline{F}_m.$$

Definition 6. A **tangent vector** v at $m \in M$ is a linear derivation of the algebra $\overline{F}_m$. That is, for all $\mathbf{f}$, $\mathbf{g} \in \overline{F}_m$, and $\lambda \in \mathbb{R}$,

$$v(\mathbf{f} + \lambda\mathbf{g}) = v(\mathbf{f}) + \lambda v(\mathbf{g}),$$

$$v(\mathbf{fg}) = \mathbf{f}(m)v(\mathbf{g}) + \mathbf{g}(m)v(\mathbf{f}).$$

Let M_m denote the set of tangent vectors to M at m. We call M_m the **tangent space** to M at m. Note that M_m is a real vector space, whose dimension is the same as the dimension of M.

Example 4. If $\mathbf{c}$ is the germ of a function with constant value c on a neighborhood of m, and if v is a tangent vector at m, then $v(\mathbf{c}) = 0$, since $v(\mathbf{c}) = cv(1)$, and $v(1) = v(1 * 1) = 1v(1) + 1v(1) = 2v(1)$.

Lemma 1. *M_m is naturally isomorphic to $(F_m/F_m^2)^*$. Here, $*$ denotes the dual vector space.*

Theorem 1. *dim $(F_m/F_m^2) =$ dim$M$.*

Theorem 2. *dim $M_m =$ dim M.*

Definition 7. Let (U, ϕ) be a coordinate system with coordinate functions $x_1, \ldots, x_d)$ and let $m \in U$. For each $i \in (1, \ldots, d)$, we define the tangent vector in M_m as the directional derivative of f at m in the direction of the x_i coordinate.

1.5 Differentials

Let $\psi : M \to N$ be C^∞, and let $m \in M$. The **differential** of ψ at m is the linear map

$$d\psi : M_m \to N_{\psi(m)}$$

definite as follows: If $v \in M_m$, then $d\psi(v)$ is a tangent vector at $\psi(m)$. Let g be a $C\infty$ function on a neighborhood of $\psi(m)$. Definite $d\psi(v)(g)$ by setting

$$d\psi(v)(g) = v(g \circ \psi).$$

One may check without much work that $d\psi$ is a linear map of M_m into $N_{\psi(m)}$. The map ψ is called **non-singular** at m if $d\psi_m$ is non-singular (in other words, the kernel of the map is 0). The **dual map**

$$\delta\psi : N^*_{\psi(m)} \to M^*_m$$

is definite as usual, under the requirement that

$$\delta\psi(\omega)(v) = \omega(d\psi(v))$$

whenever $\omega \in N^*_{\psi(m)}$ and $v \in M_m$.

Theorem 3. *Let ψ be a C^∞ mapping of the connected manifold M into the manifold N. Suppose that for each $m \in M$, $d\psi_m \equiv 0$. Then ψ is a constant mapping.*

We finish our notes in this section with a quick discussion of *tangent bundles*. It turns out that the collection of all tangent vectors to a differentiable manifold forms a differentiable manifold on its own. This manifold is the tangent bundle. See [2] for more information on tangent bundles and cotangent bundles.

2 Lie Groups

We switch our discussion to the topic of Lie groups and smooth manifolds, returning to tangent spaces for a brief moment. To finish the section, we discuss and present some examples.

2.1 Smooth Manifolds

Definition 8. A continuous map $g : V_1 \to V_2$ where each $V_k \subset \mathbb{R}^{m_k}$ is open is called **smooth** if it is infinitely differentiable. A smooth bijection g is a **diffeomorphism** if its inverse $g^{-1} : V_2 \to V_1$ is also smooth.

Definition 9. A topological space X is called **separable** if it has a countable open covering.

Note that every compact topological space is separable. In addition, any subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ is separable, too ($n \geq 1$). Thus, all matrix groups are separable. Finally, a quotient space of a separable space is also separable. In the below, let M be a separable Hausdorff topological space.

Definition 10. If $U \subset M$ and $V \subset \mathbb{R}^n$ are open subsets, a homeomorphism $f : U \to V$ is called an **n-chart** for U. If $\mathcal{U} = \{U_a | a \in A\}$ is an open covering of M and $\mathcal{F} = \{f_a : U_a \to V_a\}$ is a collection of charts, then $\mathcal{F}$ is called an **atlas** for M if, whenever $U_a \cap U_b \neq \emptyset$,

$$f_b \circ f_a^{-1} : f_a(U_a \cap U_b) \to f_b(U_a \cap U_b)$$

is a diffeomorphism.

At times, we denote the atlas by $(M, \mathcal{U}, \mathcal{F})$ and refer to it as a **smooth manifold of dimension** n or *smooth n-manifold.*

Definition 11. Let $(M, \mathcal{U}, \mathcal{F})$ and $(M', \mathcal{U}', \mathcal{F}')$ be atlases on the topological spaces M and M', respectively. A **smooth map** $h : (M, \mathcal{U}, \mathcal{F}) \to (M', \mathcal{U}', \mathcal{F}')$ is a continuous map $h : M \to M'$ such that for each pair a, a' with $h(U_a) \cap U'_a \neq \emptyset$, the composite map

$$f'_{a'} \circ h \circ f_a^{-1} : f_a(h^{-1}U'_{a'}) \to V'_{a'}$$

is smooth.

2.2 Tangent Spaces

Let $(M, \mathcal{U}, \mathcal{F})$ be a smooth n-manifold and let $p \in M$. Let $\gamma : (a,b) \to M$ be a continuous curve with $a < 0 < b$.

Definition 12. γ is **differentiable** at $t \in (a,b)$ if for every chart $f : U \to V$ with $\gamma(t) \in U$, the curve $f \circ \gamma : (a,b) \to V$ is differentiable at $t \in (a,b)$. That is, $(f \circ \gamma)'(t)$ exists. γ is **smooth** at $t \in (a,b)$ if all the derivatives of $f \circ \gamma$ exist at t.

Lemma 2. *Let $f_0 : U_0 \to V_0$ be a chart with $\gamma(t) \in U_0$ and suppose that*

$$f_0 \circ \gamma : (a,b) \cap f_0^{-1} V_0 \to V_0$$

is differentiable at t. Then for any chart $f : U \to V$ with $\gamma(t) \in U$,

$$f \circ \gamma : (a,b) \cap f^{-1} V \to V$$

is differentiable at t.

Theorem 4. *Let $p \in M$, and let T_pM denote the **tangent space** to the manifold M at p. Then T_pM is an $\mathbb{R}$-vector space of dimension n.*

Definition 13. Let $(M, \mathcal{U}, \mathcal{F})$ be a manifold of dimension n. A subset $N \subset M$ is a **submanifold** of dimension k if for every $p \in N$ there is an open neighborhood $U \subset M$ of p and an n-chart $f : U \to V$ such that

$$p \in f^{-1}(V \cap \mathbb{R}^k) = N \cap U.$$

For such N we form k-charts of the form

$$f_0 : N \cap U \to V \cap R_k,$$

such that $f_0(x) = f(x)$. Denote this manifold by $(N, \mathcal{U}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}})$. We then have the following result.

Theorem 5. *For a submanifold $N \subset M$ of dimension k, the inclusion function $i : N \to M$ is smooth, and for every $p \in N$, $di_p : T_pN \to T_pM$ is an injection*

Theorem 6 (Implicit Function Theorem for Manifolds)**.** *Let $h : (M, \mathcal{U}, \mathcal{F}) \to (M', \mathcal{U}', \mathcal{F}')$ be a smooth map between manifolds of respective dimensions n and n'. Suppose that for some $q \in M'$, $dh_p : T_pM \to T_{h(p)}M'$ is surjective for every $p \in N = h^{-1}q$. Then $N \subset M$ is a submanifold of dimension $n - n'$, and the tangent space at $p \in N$ is given by $T_pN = ker\ h_p$.*

Theorem 7 (Inverse Function Theorem for Manifolds)**.** *Let $h : (M, \mathcal{U}, \mathcal{F}) \to (M', \mathcal{U}', \mathcal{F}')$ be a smooth map between manifolds of respective dimensions n and n'. Suppose that for some $p \in M$, $dh_p : T_pM \to T_{h(p)}M'$ is an* *isomorphism. Then there is an open neighborhood $U \subset M$ of p and an open neighborhood $V \subset M'$ of $h(p)$ such that $hU = V$, and the restriction of h to the map $h_1 : U \to V$ is a diffeomorphism. Particularly, the derivative $dh_p : T_p \to T_{h(p)}$ is an $\mathbb{R}$-linear isomorphism, and $n = n'$.*

When we have the above situation, we say that h is **locally a diffeomorphism** at p.

2.3 Lie Groups

Definition 14. Let G be a smooth manifold that is also a topological group with multiplication map $mult : G \times G \to G$ and inverse map $inv : G \to G$. View $G \times G$ as the product manifold. Then G is a **Lie group** if $mult$ and inv are smooth maps.

Definition 15. Let G be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a **Lie subgroup** of G. We then have that the restriction to G of the multiplication and inverse maps on G are smooth. Thus, H is also a Lie group.

Let G be a Lie group, and let $g \in G$. For g there is a tangent space T_pG. Let $\mathcal{G} = T_1G$ be the tangent space at the identity of G. A smooth homomorphism of Lie groups $G \to H$ has the properties of a Lie homomorphism.

Consider the following basic yet important properties for $g \in G$:

$$\begin{aligned} L_g : G \to G; &\quad L_g(x) = gx, \\ R_g : G \to G; &\quad R_g(x) = xg, \\ \xi : G \to G; &\quad \xi(x) = gxg^{-1}. \end{aligned}$$

The above are the properties of left multiplication, right multiplication, and conjugation, respectively.

Theorem 8. *For each $g \in G$, the maps L_g, R_g, and ξ_g are diffeomorphisms with inverses*

$$L_g^{-1} = L_{g^{-1}}, \quad R_g^{-1} = R_{g^{-1}}, \quad \xi_g^{-1} = \xi_{g^{-1}}.$$

Before getting to some examples, let us consider the derivatives of the above maps at the identity of G, 1. As L_g and R_g are diffeomorphisms with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, the derivatives

$$d(L_g)_1, d(R_g)_1 : \mathcal{G} : T_1G \to T_gG$$

are $\mathbb{R}$-linear isomorphisms. We use this property to identify every tangent space of G with $\mathcal{G}$. The conjugation map ξ_g fixes the identity and thus induces an $\mathbb{R}$-linear isomorphism

$$Ad_g = d(\xi_g)_1 : \mathcal{G} \to \mathcal{G}.$$

The above is the **adjoint action** of $g \in G$ on $\mathcal{G}$. The natural Lie bracket $[,]$ on $\mathcal{G}$ makes $\mathcal{G}$ into an $\mathcal{R}$-Lie algebra.

Theorem 9. *Let G and H be Lie groups, and let $\phi : G \to H$ be a Lie homomorphism. Then the derivative is a homomorphism of Lie algebras. In particular, if $G \leq H$ is a Lie subgroup, the inclusion map $i : G \to H$ induces an injection of Lie algebras $di : \mathcal{G} \to \mathcal{H}$.*

2.4 Examples

Before moving on to our notes on maximal tori and compact connected Lie groups, we present some examples to help solidify understanding.

Example 5. When $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, $GL_n(\mathbf{k})$ is a Lie group: It is true that $GL_n(\mathbf{k}) \subset M_n(\mathbf{k})$ is an open subset. For charts we take the open sets $U \subset GL_n(\mathbf{k})$ and the identity function $id : U \to U$. The tangent space at each point $A \in GL_n(\mathbf{k})$ is $M_n(\mathbf{k})$. The multiplication and inverse maps are both smooth, as they are defined by polynomial and rational functions between subsets of $M_n(\mathbf{k})$.

Example 6. For $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, $SL_n(\mathbf{k}) \leq GL_n(\mathbf{k})$ is a Lie subgroup.

Theorem 10 (Left Translation Trick)**.** *Let $F : GL_n(\mathbb{R}) \to M$ be a smooth function into a smooth manifold M. Also, suppose that $B \in GL_n(\mathbb{R})$ satisfies $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. Let $A \in GL_n(\mathbb{R})$ with dF_A surjective. Then dF_{BA} is also surjective.*

Theorem 11 (Identity Check Trick)**.** *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup and M be a smooth manifold. Let $F : GL_n(\mathbb{R}) \to M$ be a smooth manifold with $F^{-1}q = G$ for some $q \in M$. Also, suppose that for every $B \in G$, $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. If dF_1 is surjective, then dF_A is surjective for all $A \in G$ and ker $dF_A = A\mathcal{G}$.*

Example 7. $O(n)$ is a Lie subgroup of $GL_n(\mathbb{R})$.

Theorem 12. *Let $G \leq GL_n(\mathbb{R})$ be a matrix group which is also a submanifold (and therefore a Lie subgroup). Then the tangent space to G at I agrees with the Lie algebra $\mathcal{G}$, and the dimension of the smooth manifold G is dim G. In general, $T_A G = A\mathcal{G}$.*

Theorem 13. *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup. Then G is a Lie subgroup of $GL_n(\mathbb{R})$.*

Theorem 14. *Let $G \leq H$ be a closed subgroup of a Lie group H. Then G is a Lie subgroup of H.*

3 Maximal Tori in Compact Connected Lie Groups

We turn the remainder of our discussion over to maximal tori, which turn out to be very important in the classification of simple compact Lie groups. First: A theorem that will help us on our way.

Theorem 15. *Let G be a compact Lie group. Then for some $m, n \geq 1$, there are injective Lie homomorphisms $G \to O(m)$ and $G \to U(n)$. Therefore, G is a matrix group.*

3.1 Introducing Tori

Definition 16. The **circle group** is defined to be $T = \{z \in \mathbb{C} || z| = 1\} \leq \mathbb{C}^x$. It turns out that the circle group is also a matrix group – and a closed and bounded subset of $\mathbb{C}$. Hence, the circle group is also compact, path-connected, and abelian.

Definition 17. For each $r \geq 1$,

$$T^r = \{diag(z_1, \ldots, z_r) : |z_1| = \cdots = |z_r| = 1\} \leq GL_r(\mathbb{C})$$

is the **standard torus of rank** r. More generally, a torus of rank r is any Lie group isomorphic to T^r.

Theorem 16. *Let T be a torus. Then T is a compact, path-connected, abelian Lie group.*

We see from the next result which compact Lie groups can be classified as tori.

Theorem 17. *Let H be a compact Lie group. Then H is a torus if and only if it is connected and abelian.*

Theorem 18. *Let T be a torus of rank r. Then the exponential map exp: $t \to T$ is a surjective homomorphism of Lie groups whose kernel is a discrete subgroup isomorphic to $\mathbb{Z}^r$. Therefore, there is an isomorphism of Lie groups $\mathbb{R}^r/\mathbb{Z}^r \simeq T$.*

Definition 18. Let G be a Lie group. Then an element $g \in G$ is called a **topological generator** (or just a generator) of G if the cyclic subgroup $< g > \leq$ is dense in G. That is, $\overline{< g >} = G$.

Theorem 19. *Every torus has a generator.*

3.2 Maximal Tori in Compact Lie Groups

Definition 19. Let G be a Lie group and let $T \leq G$ be a closed subgroup which is a torus. Then T is **maximal** in G if the only torus $T' \leq G$ for which $T \leq T'$ is T.

Note that it is clear that every torus $T \leq G$ is contained in some maximal torus (contained in G). It is also the case that if G is connected, then every element $g \in G$ is also contained in a maximal torus. See Baker ([1]) for a list of examples of maximal tori in well-known groups.

Now suppose that G is a compact connected Lie group, and let $T \leq G$ be a maximal torus.

Theorem 20. *If $g \in G$, then there is an $x \in G$ such that $g = xTx^{-1}$. In other words, g is conjugate to some element of T. Then we may write G as a union of such conjugates.*

Theorem 21. *If $T and T'$ are two maximal tori in G, then they are conjugate in G.*

Theorem 22. *Suppose that $g \in G$ and $H, K \leq G$ are Lie subgroups with Lie algebras $\mathcal{H}, \mathcal{L} \leq \mathcal{G}$. If $gHg^{-1} = K$, then $Ad_{\mathcal{G}}\mathcal{H} = \mathcal{L}$.*

Theorem 23. *Let G be a compact connected Lie group. Then its exponential map is surjective.*

3.3 Weyl Groups and Normalizers

We end our notes with several theorems and definitions for both reference and guidance (background) for future studies.

Theorem 24. *Let $A \leq G$ be a compact abelian Lie group, and suppose that $A_1 \leq A$ is the connected component of the identity element. If A/A_1 is cyclic, then A has a generator. Hence, A is contained in a torus in G.*

Theorem 25. *Let $A \leq G$ be a connected abelian subgroup, and let $g \in G$ commute with all the elements of A. Then there is a torus $T \leq G$ that contains the subgroup $(A, g) \leq G$ that is generated by A and g.*

Theorem 26. *Let $T \leq G$ be a maximal torus, and let $T \leq A \leq G$, where A is abelian. Then $A = T$. In other words, every maximal torus is a maximal abelian subgroup.*

As we have now established that every maximal torus is also a maximal abelian subgroup, let us consider the relationship that a maximal torus T has with the rest of G.

Lemma 3. *Let $T \leq G$ be a torus, and let $Q \leq N_G(T)$ be a connected subgroup acting on T by conjugation. Then Q acts trivially. That is, for all $g \in Q$ and $x \in T$,*

$$gx = gxg^{-1} = x.$$

Theorem 27. *Let $T \leq G$ be a maximal torus. Then the Weyl group $W_G(T)$ is isomorphic to the group of components of the normalizer and therefore is finite. Also, $W_G(T)$ acts on T by conjugation.*

Theorem 28. *Let $T \leq G$ be a maximal torus, and let $x, y \in T$. If x and y are conjugate in G, then they are conjugate in $N_G(T)$. Therefore, there is some element $w \in W_G(T)$ for which $y = wx$.*

References

[1] Baker, Andrew., *Matrix Groups: An Introduction to Lie Group Theory*, Great Britain, Springer, 2001.

[2] Warner, Frank W., *Foundations of Differentiable Manifolds and Lie Groups*, London, Scott, Foresman and Company, 1970(?).