High-Order Perturbation of Surfaces Method for Bounded-Obstacle Scattering in Polar Coordinates

Xin Tong

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

Lawrence berkeley national laboratory, June 2016

Outline

Maxwell's Equations

The governing equations are the Time-Harmonic Maxwell's Equations in a homogeneous region

$$\begin{cases} \nabla \times \mathbf{E} &= i\omega\mu_0 \mathbf{H} \\ \nabla \times \mathbf{H} &= -i\omega\epsilon_0\epsilon \mathbf{E} \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \end{cases}$$

- **E**-electric field, **H**-magnetic field
- There is no free charge.
- The complex permittivity is defined by $\epsilon := \epsilon' + i\sigma/(\omega\epsilon_0)$
- The σ is conductivity.

Two-Dimensional Simplifications

- The grating shape is invariant in the θ_2-direction:

$$r = a + g(\theta_1)$$

- In this 2-D setting Maxwell's Equations we consider Transverse Electric (TE) and Transverse Magnetic (TM) polarizations.
- Boundary conditions: at any material interface we enforce tangential continuity of **E** and **H**

$$\mathbf{N} \times \mathbf{E} = 0, \quad \mathbf{N} \times \mathbf{H} = 0,$$

  where **N** is a normal to the interface.
- Incident, scattered, total fields are all 2π-periodic.

Governing Equation for Doubly-layered medium

We seek periodic solutions of

$$\begin{cases} \Delta u + k_u^2 u = 0 & r > a + g(\theta) \\ \Delta w + k_w^2 w = 0 & r < a + g(\theta) \\ u - w = -u^i & r = a + g(\theta) \\ \partial_{\mathbf{N}} u - \tau^2 \partial_{\mathbf{N}} w = -\partial_{\mathbf{N}} u^i & r = a + g(\theta) \end{cases}$$

- u^i is the incident radiation.
- $\mathbf{N} = (a + g, -g')$
- $\tau^2 = 1$ in TE, $\tau^2 = (k_u/k_w)^2$ in TM.

Governing Equation

We are solving the Helmholtz equation on a two-dimensional domain exterior to a bounded obstacle:

$$\Delta u + k^2 u = 0 \qquad r > a + g(\theta) \qquad (1')$$

$$u(r,\theta) = \xi(\theta) \qquad r = a + g(\theta) \qquad (2')$$

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - iku) = 0 \qquad r \to \infty \qquad (3')$$

- The solution must satisfy (3') the Sommerfeld radiation condition (SRC) to guarantee a physical solution.
- Let $b > a + |g|_{L^\infty}$, then using method of separation of variables gives the general solution $u(r,\theta)$ of (1') and (3') for $r > b$.

General Solution

Considering the bounded domain $\{(r,\theta) : a + g(\theta) < r < b\}$ instead of the former unbounded domain $\{(r,\theta) : r > a + g(\theta)\}$, we rewrite our governing equation as

$$\Delta u + k^2 u = 0 \qquad a + g(\theta) < r < b \qquad (1)$$

$$u(r,\theta) = \xi(\theta) \qquad r = a + g(\theta) \qquad (2)$$

$$\partial_r u(b,\theta) - Tu(b,\theta) = 0 \qquad r = b \qquad (3)$$

where we define an operator T by $T(u(b,\theta)) := \partial_r u(b,\theta)$.

- The solution to (1) and (3) is

$$u(r,\theta) = \sum_p a_p \frac{H_p^{(1)}(kr)}{H_p^{(1)}(ka)} e^{ip\theta}$$

Field Expansion 1

- Suppose u depends *analytically* upon ε so that we can write Taylor expansion of u and the series converges in a proper function space:

$$u = u(r,\theta) = u(r,\theta;\varepsilon) = \sum_{n=0}^{\infty} u_n(r,\theta)\varepsilon^n$$

- Plug the $u(r,\theta;\varepsilon)$ into the governing equations and find equation on each order of n. The solution u is of the form

$$u(r,\theta;\varepsilon) = \sum_{n=0}^{\infty} \sum_{p} a_{n,p} \frac{H_p^{(1)}(kr)}{H_p^{(1)}(ka)} e^{ip\theta} \varepsilon^n$$

- We are looking for the coefficients $\{a_{n,p}\}$ using the boundary data $\xi(\theta)$ and let $g(\theta) = \varepsilon f(\theta)$.

Field Expansion 2

- Define $a_n(\theta)$ and $a(\theta)$ by

$$\begin{cases} \text{sum in } p: & a_n(\theta) := \sum_p a_{n,p} e^{ip\theta} \\ \text{sum in } n: & a(\theta) := \sum_{n=0}^{\infty} a_n \varepsilon^n = u(a,\theta) \end{cases}$$

- Define the 'zero-trace' to 'boundary-trace' operator $\mathcal{D}$ by

$$\mathcal{D} : u(a,\theta) \to u(a + \varepsilon f(\theta), \theta)$$

Then the equation (2) $u(a + \varepsilon f(\theta), \theta) = \xi(\theta)$ is expressed as

$$\mathcal{D}(a(\theta)) = \xi(\theta)$$

- HOPS scheme

$$\left[\sum_{n=0}^{\infty} \mathcal{D}_n \varepsilon^n\right] \left[\sum_{n=0}^{\infty} a_n \varepsilon^n\right] = \sum_{n=0}^{\infty} \xi_n \varepsilon^n$$

Field Expansion 3

At each n, we have

$$n = 0 \qquad \mathcal{D}_0 a_0 = \xi_0$$

$$n \geq 1 \qquad \mathcal{D}_0 a_n = \xi_n - \sum_{m=0}^{n-1} \mathcal{D}_{n-m} a_m$$

where

$$\mathcal{D}_0 = \mathbf{I}, \quad \mathcal{D}_0^{-1} = \mathbf{I}$$

$$\mathcal{D}_n[e^{ip\theta}] = k^n F_n \frac{d_z^n H_p^{(1)}(ka)}{H_p^{(1)}(ka)} e^{ip\theta}, \quad F_n = F_n(\theta) = \frac{f(\theta)^n}{n!}.$$

At each wave number p, we can solve (in Fourier space) for $\{a_{n,p}\}$

$$a_{0,p} = (\hat{\xi}_0)_p$$

$$a_{n,p} = (\hat{\xi}_n)_p - \sum_{m=0}^{n-1} k^{n-m} \sum_q (\hat{F}_{n-m})_{p-q} \frac{d_z^{n-m} H_p^{(1)}(ka)}{H_p^{(1)}(ka)} a_{m,q}$$

Application: Dirichlet-to-Neumann Operator

We will use the coefficients $\{a_{n,p}\}$ to approximate the exterior Neuman boundary condition $\nu(\theta)$:

$$\begin{aligned}\nu(\theta) :&= \left[-\partial_{\mathbf{N}} u\right](a+\varepsilon f, \theta) = \left[-\mathbf{N}\cdot\nabla u\right](a+\varepsilon f, \theta) \\ &= -(a+\varepsilon f)\partial_r u(a+\varepsilon f, \theta) + \frac{1}{a+\varepsilon f}\varepsilon(\partial_\theta f)\partial_\theta u(a+\varepsilon f, \theta)\end{aligned}$$

- Define the Dirichlet-to-Neumann operator by $\mathbf{G}(g) : \xi \to \nu$ such that $\mathbf{G}(g)\xi = \nu$
- HOPS Scheme: $\mathbf{G}(\varepsilon f)\xi = \sum_{n=0}^{\infty} G_n \xi \varepsilon^n = \nu$. Next, evaluate $u(r,\theta) = \sum_{n=0}^{\infty}\sum_p a_{n,p}\frac{H_p^{(1)}(kr)}{H_p^{(1)}(ka)}e^{ip\theta}\varepsilon^n$ at $r = a+\varepsilon f$ then compute $\nu(\theta)$ by definition.

Application: Dirichlet-to-Neumann Operator

For each n, we can get $G_n\xi$ in terms of $\{a_{n,p}\}$.

$$G_0\xi = -ak\sum_p a_{0,p}\frac{d_z H_p^{(1)}(ka)}{H_p^{(1)}(ka)}e^{ip\theta}$$

$$G_n\xi = -\frac{f}{a}G_{n-1}\xi - a\sum_{m=0}^{n}\sum_p a_{m,p}k^{n-m+1}F_{n-m}(\theta)\frac{d_z^{n-m+1}H_p^{(1)}(ka)}{H_p^{(1)}(ka)}e^{ip\theta}$$

$$-2f\sum_{m=0}^{n-1}\sum_p a_{m,p}k^{n-m}F_{n-m-1}(\theta)\frac{d_z^{n-m}H_p^{(1)}(ka)}{H_p^{(1)}(ka)}e^{ip\theta}$$

$$-\frac{f^2}{a}\sum_{m=0}^{n-2}\sum_p a_{m,p}k^{n-m-1}F_{n-m-2}(\theta)\frac{d_z^{n-m-1}H_p^{(1)}(ka)}{H_p^{(1)}(ka)}e^{ip\theta}$$

$$+\frac{\partial_\theta f}{a}\sum_{m=0}^{n-1}\sum_p a_{m,p}k^{n-m-1}F_{n-m-1}(\theta)\frac{d_z^{n-m-1}H_p^{(1)}(ka)}{H_p^{(1)}(ka)}e^{ip\theta}$$

Numerical Approach

We choose the parameters:

- $k = 1, f = e^{\cos(\theta)}$
- $\xi(\theta) = [H_p^{(1)}(kr)e^{ip\theta}](r = a + \varepsilon f)$ for any wave number p
- exact solution: $u(r, \theta) = H_p^{(1)}(kr)e^{ip\theta}$

List of all Matlab files:

- test_helmholtz_polar.m
- field_fe_helmholtz_polar.m
- dno_fe_helmholtz_polar.m
- compute_errors_2d_polar.m
- make_plots_polar.m
- diff_besselh.m

Governing Equation and General Solutions

The Helmholtz equation on a two-dimensional domain interior to a bounded obstacle:

$$\Delta w + k^2 w = 0 \qquad r < a + g(\theta) \tag{4}$$

$$w(r,\theta) = \xi(\theta) \qquad r = a + g(\theta) \tag{5}$$

$$\lim_{r\to 0} w(r,\theta) \text{ is bounded} \qquad r \to 0 \tag{6}$$

The solution to (4) and (6) is

$$w(r,\theta) = \sum_p d_p \frac{J_p(kr)}{J_p(ka)} e^{ip\theta}$$

Field Expansion

Following steps of field expansion above, we can also get the coefficients $\{d_{n,p}\}$ at each wave number p (in Fourier space)

$$d_{0,p} = (\hat{\xi}_0)_p$$

$$d_{n,p} = (\hat{\xi}_n)_p - \sum_{m=0}^{n-1} k^{n-m} \sum_q (\hat{F}_{n-m})_{p-q} \frac{d_z^{n-m} J_p(ka)}{J_p(ka)} a_{m,q}$$

Application: Dirichlet-to-Neumann Operator

The idea here are similar the exterior domain and the only difference is the sign

$$\begin{aligned}\nu(\theta) :&= \left[+\partial_{\mathbf{N}} w\right](a+\varepsilon f, \theta) = \left[+\mathbf{N} \cdot \nabla w\right](a+\varepsilon f, \theta) \\ &= (a+\varepsilon f)\partial_r w(a+\varepsilon f, \theta) - \frac{1}{a+\varepsilon f}\varepsilon(\partial_\theta f)\partial_\theta w(a+\varepsilon f, \theta)\end{aligned}$$

- Define the Dirichlet-to-Neumann operator by $\mathbf{G}(g) : \xi \to \nu$ such that $\mathbf{G}(g)\xi = \nu$

Application: Dirichlet-to-Neumann Operator

For each n, we can also get $G_n\xi$ in terms of $\{d_{n,p}\}$.

$$G_0\xi = ak\sum_p d_{0,p}\frac{d_z J_p(ka)}{J_p(ka)}e^{ip\theta}$$

$$\begin{aligned}
G_n\xi = &-\frac{f}{a}G_{n-1}\xi + a\sum_{m=0}^{n}\sum_p d_{m,p}k^{n-m+1}F_{n-m}(\theta)\frac{d_z^{n-m+1}J_p(ka)}{J_p(ka)}e^{ip\theta}\\
&+2f\sum_{m=0}^{n-1}\sum_p d_{m,p}k^{n-m}F_{n-m-1}(\theta)\frac{d_z^{n-m}J_p(ka)}{J_p(ka)}e^{ip\theta}\\
&+\frac{f^2}{a}\sum_{m=0}^{n-2}\sum_p d_{m,p}k^{n-m-1}F_{n-m-2}(\theta)\frac{d_z^{n-m-1}J_p(ka)}{J_p(ka)}e^{ip\theta}\\
&-\frac{\partial_\theta f}{a}\sum_{m=0}^{n-1}\sum_p d_{m,p}k^{n-m-1}F_{n-m-1}(\theta)\frac{d_z^{n-m-1}J_p(ka)}{J_p(ka)}e^{ip\theta}
\end{aligned}$$

Numerical Approach

We choose the parameters:

- $k = 1, f = e^{\cos(\theta)}$
- $\xi(\theta) = [J_p(kr)e^{ip\theta}](r = a + \varepsilon f)$ for any wave number p
- exact solution: $w(r, \theta) = J_p(kr)e^{ip\theta}$

List of all Matlab files:

- test_helmholtz_polar_interior.m
- field_fe_helmholtz_polar_interior.m
- dno_fe_helmholtz_polar_interior.m
- compute_errors_2d_polar.m
- make_plots_polar.m
- diff_besselj.m

Future work: Doubly-layered medium

We seek periodic solutions of

$$\begin{cases} \Delta u + k_u^2 u = 0 & r > a + g(\theta) \\ \Delta w + k_w^2 w = 0 & r < a + g(\theta) \\ u - w = -u^i & r = a + g(\theta) \\ \partial_{\mathbf{N}} u - \tau^2 \partial_{\mathbf{N}} w = -\partial_{\mathbf{N}} u^i & r = a + g(\theta) \end{cases}$$

- Define $\zeta(\theta) = -u^i|_{r=a+g(\theta)}$ and $\psi(\theta) = -\partial_{\mathbf{N}} u^i|_{r=a+g(\theta)}$
- the boundary conditions can be expressed as

$$\begin{aligned} u - w &= \zeta(\theta) & r &= a + g(\theta) \\ \mathbf{G}^u u + \tau^2 \mathbf{G}^w w &= -\psi(\theta) & r &= a + g(\theta) \end{aligned}$$

- Use $\mathbf{G}^u$ and $\mathbf{G}^w$ (known) to find the solution.

Thank you!

Comments and Questions!

References I

- J. Billingham and A. King, *Wave Motion*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2000.
- D. P. Nicholls and J. Shen, *A stable high-order method for two-dimensional bounded-obstacle scattering*, SIAM Journal Scientific Computing, 28 (2006), pp. 1398–1419.