A Brief Introduction to GAIL (Guaranteed Automatic Integration Library) Version 2.3

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Overview

GAIL algorithms compute answers of guaranteed accuracy for multidimensional integration as well as univariate integration, function approximation and optimization:

- Requires only input function and error tolerance
- Theoretically sound stopping criterion
- Well tested, documented, free, and open-source

Introduction

- Designing algorithms for a *cone* of input functions allows us to prove that our stopping criteria are valid and information cost is optimal.
- What is not observed about the function is not much worse than what is observed.
- We follow the philosophy of reproducible research & sustainable practices of software development.

Approximation & Optimization

❶ `funappx_g`: One-dimensional function approximation on bounded interval [1]

❷ `funmin_g`: Global minimum value of univariate function on a closed interval [1]

Option Pricing

❶ `assetPath`: A class of discretized stochastic processes that model the values of an asset with respect to time [2]

❷ `optPayoff`: A class of option payoffs based on asset paths [2]

❸ `optPrice`: A class that computes the price of an option via (quasi-)Monte Carlo methods [2]

Integration

❶ `integral_g`: One-dimensional integration on bounded interval [2]

❷ `meanMC_g`: Monte Carlo (MC) method for estimating mean of a random variable [3]

❸ `cubMC_g`: MC method for multiple integration [3]

❹ `cubSobol_g`: Quasi-Monte Carlo (QMC) method using Sobol' cubature for multiple integration [4]

❺ `cubLattice_g`: QMC method using rank-1 lattices cubature for multiple integration [5]

❻ `cubBayesLattice_g`: Bayesian cubature method using lattice sampling for multiple integration [6]

❼ `meanMC_CLT`: MC method with Central Limit Theorem (CLT) confidence intervals for estimating mean of a random variable [2]

Example 1

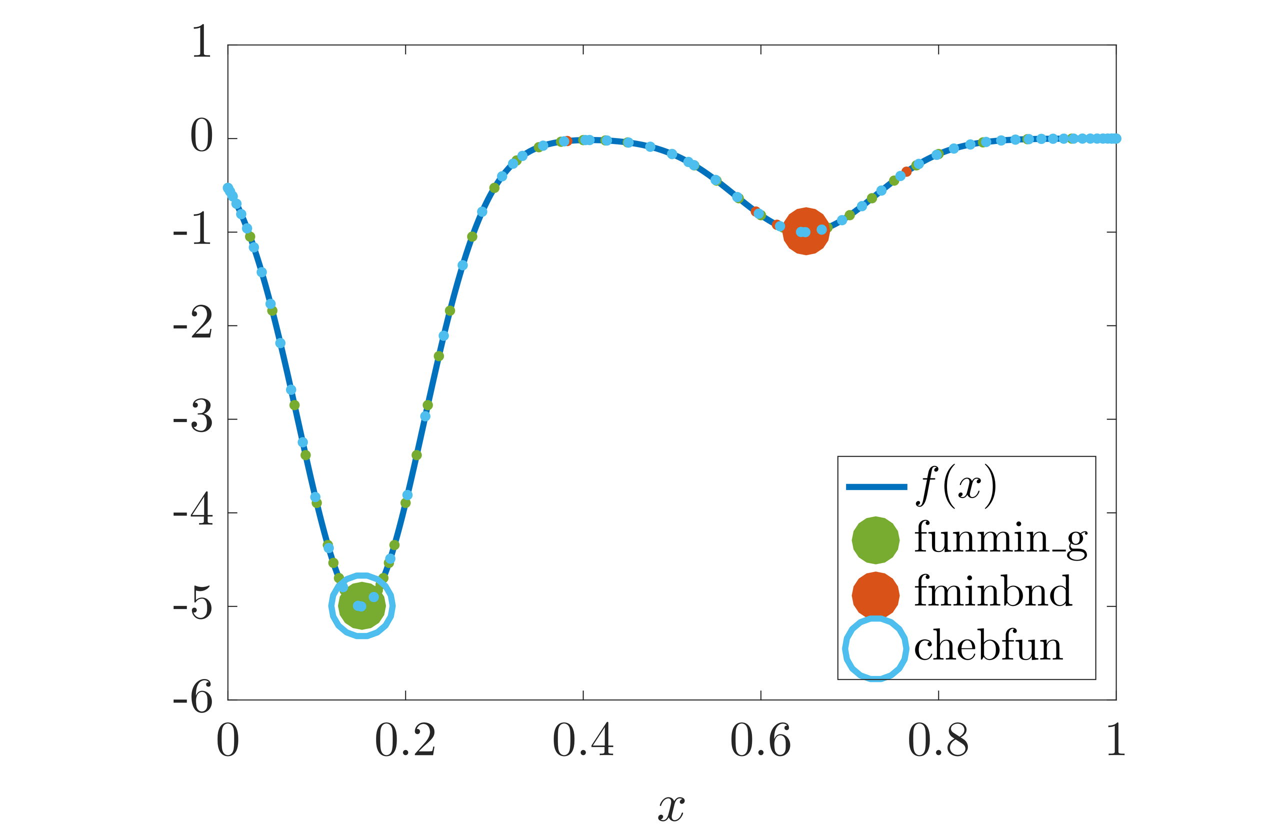

Figure 1: We want to find the global minimum of $f(x) = -5e^{-100(x-0.2)^2} - e^{-100(x-1)^2}$ for $x \in [0, 1.5]$. Our `funmin_g` locates it but Matlab's `fminbnd` returns a local minimum. Our algorithm automatically samples the function more often in spiky areas.

Example 2

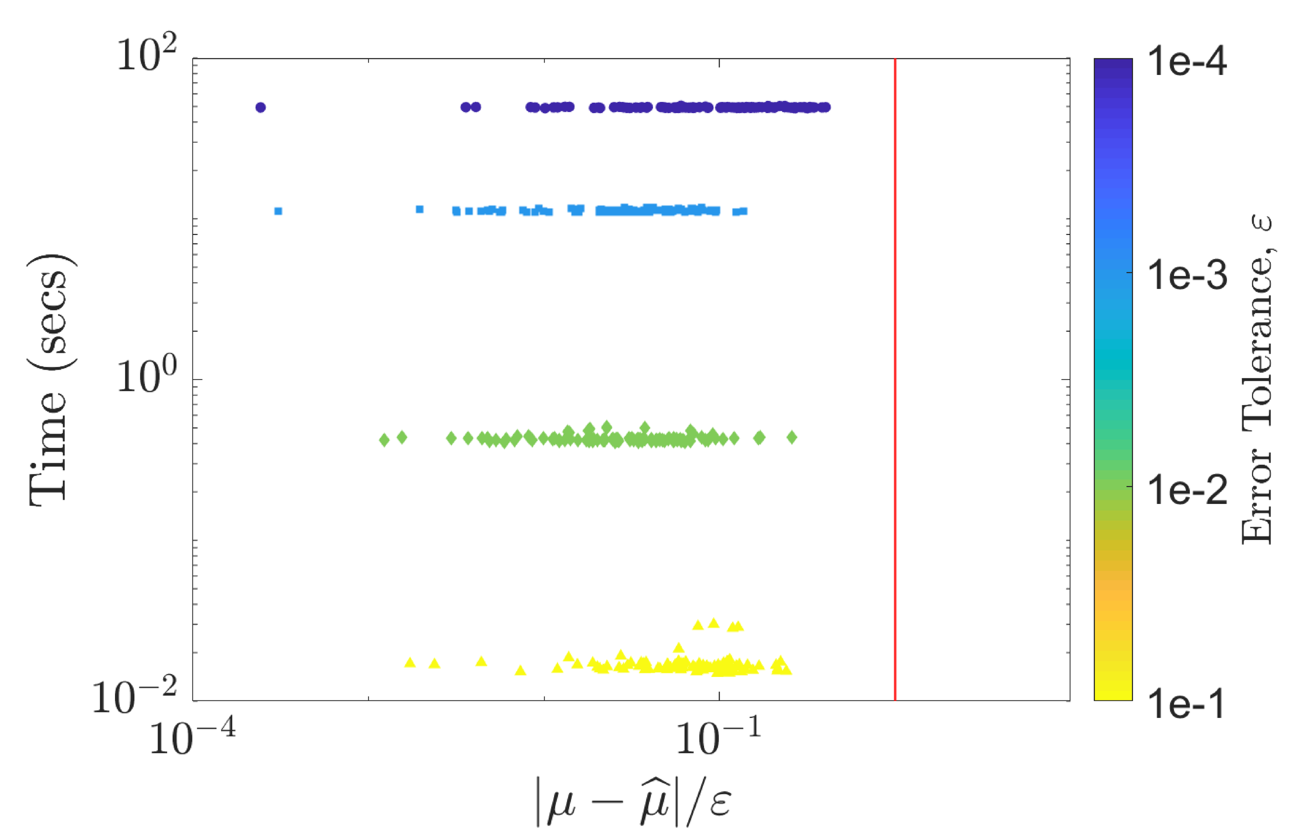

Figure 2: Pricing arithmetic mean Asian call option by `cubBayesLattice_g` with equal initial stock price and strike price $S_0 = K = 100$, maturity $T = 1/4$, risk-free interest rate $r = 5\%$, integral dimension $d = 13$, and volatility $\sigma = 0.5$. The tolerances are $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$. A sufficiently large sample size is chosen automatically to satisfy each ϵ. The success rate is 100% for this example.

Example 3

Table 1: Average performance of (quasi-)Monte Carlo algorithms in GAIL with automatic stopping criteria for estimating the Keister integrals [7] of dimension d for 1000 independent runs.

$d = 3,\ \varepsilon = 0.005$

Method	MC	Lattice	Sobol	Bayes
Absolute Error	0.001 20	0.000 51	0.000 53	0.000 43
Tolerance Met	100%	100%	100%	100%
n	2 500 000	4100	3900	1000
Time (seconds)	0.1400	0.0064	0.0034	0.0017

$d = 8,\ \varepsilon = 0.050$

Method	MC	Lattice	Sobol	Bayes
Absolute Error	0.012 00	0.015 00	0.007 10	0.001 70
Tolerance Met	100%	99%	100%	100%
n	7 400 000	15 000	16 000	66 000
Time (seconds)	0.8800	0.0240	0.0130	0.1700

Ongoing Work

- Submit GAIL to the Journal of Open Source Software
- Make GAIL part of the multi-research group Quasi-Monte Carlo Community Software

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