

1.2 Solutions and Initial Value Problems

1. Determine whether the given function is a solution to the given differential equation.

$$y = \sin x + x^2, \quad \frac{d^2y}{dx^2} + y = x^2 + 2$$

$$\left. \begin{array}{l} \frac{dy}{dx} = \cos x + 2x \\ \frac{d^2y}{dx^2} = -\sin x + 2 \end{array} \right\} \text{Eqn : } \begin{array}{l} \text{LHS} = (-\sin x + 2) + (\sin x + x^2) = 2 + x^2 = \text{RHS} \\ \checkmark \end{array}$$

Yes.

2. Determine whether the given relation is an implicit solution to the given differential equation.

$$e^{xy} + y = x - 1, \quad \frac{dy}{dx} = \frac{e^{-xy} - y}{e^{-xy} + x}$$

Differentiate the Eqn chain rule

$$\frac{d}{dx}(e^{xy} + y = x - 1) \Rightarrow e^{xy}(y + xy') + y' = 1 \leftarrow \text{product rule}$$

$$\text{Solve for } y' \Rightarrow y' = \frac{dy}{dx} = \frac{1 - e^{xy}y}{xe^{xy} + 1} \cdot \frac{1}{e^{xy}} = \frac{e^{-xy} - y}{e^{-xy} + x}$$

3. Verify that the function $\phi(x) = c_1 e^x + c_2 e^{-2x}$ is a solution to the linear equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2 . Determine c_1 and c_2 such that each of the following initial conditions is satisfied.

$$\left. \begin{array}{l} \phi' = c_1 e^x - 2c_2 e^{-2x} \\ \phi'' = c_1 e^x + 4c_2 e^{-2x} \end{array} \right\} \text{Eqn : } \begin{array}{l} (c_1 e^x + 4c_2 e^{-2x}) + (c_1 e^x - 2c_2 e^{-2x}) - 2(c_1 e^x + c_2 e^{-2x}) \\ = (c_1 + c_1 - 2c_1)e^x + (4c_2 - 2c_2 - 2c_2)e^{-2x} = 0 \end{array} \checkmark$$

(a) $y(0) = 2, y'(0) = 1$

c_1, c_2 must satisfy :

$$\left. \begin{array}{l} \phi(0) = c_1 + c_2 = 2 \\ \phi'(0) = c_1 - 2c_2 = 1 \end{array} \right.$$

Solve for $c_1 = \frac{5}{3}$

$$c_2 = \frac{1}{3}$$

(b) $y(1) = 1, y'(1) = 0$

$$\left. \begin{array}{l} \phi(1) = c_1 e + c_2 e^{-2} = 1 \\ \phi'(1) = c_1 e - 2c_2 e^{-2} = 0 \end{array} \right.$$

Solve for $c_1 = \frac{2}{3}e^{-1}$

$$c_2 = \frac{1}{3}e^{-2}$$

4. Determine whether the Theorem of Existence and Uniqueness of Solutions applies.

$$\frac{dy}{dt} - ty = \sin^2 t, \quad y(\pi) = 5$$

Rewrite : $\frac{dy}{dt} = ty + \sin^2 t \stackrel{\text{define}}{=} f(t, y)$

$f(t, y)$ is continuous & $\frac{\partial f}{\partial y} = t$ is continuous : So the theorem applies to any initial condition. For $y(\pi) = 5$, it applies ?

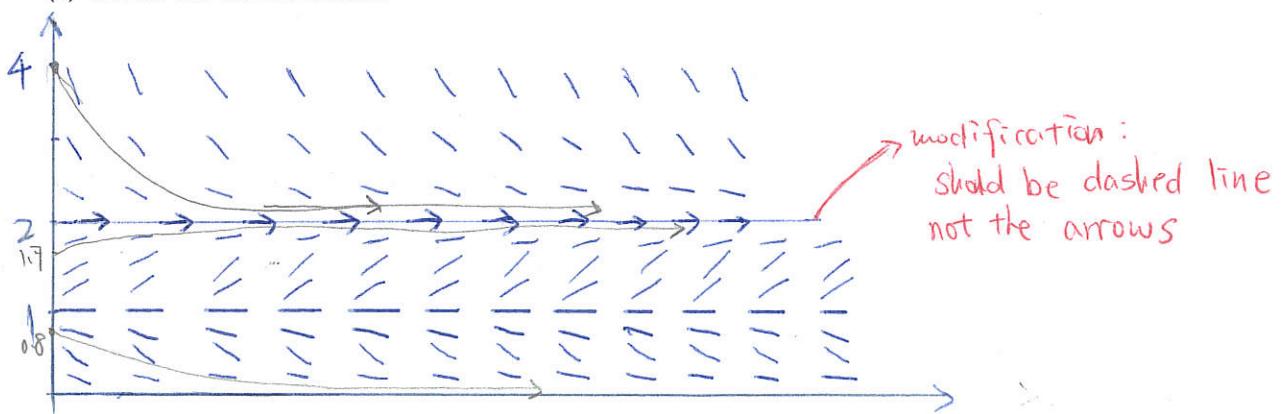
1.3 Direction Fields

1. Consider the differential equation

$$\frac{dp}{dt} = p(p-1)(2-p)$$

for the population p (in thousands) of a certain species at time t .

- (a) Sketch the direction field.



- (b) If the initial population is 4000, what is $\lim_{t \rightarrow +\infty} p(t)$?

From $p(0)=4$, the direction field has negative slope which implies the solution curve will approach $P=2$

- (c) If $p(0) = 1.7$, what is $\lim_{t \rightarrow +\infty} p(t)$?

$$\lim_{t \rightarrow \infty} p(t) = 2$$

increases to $p=2$

- (d) If $p(0) = 0.8$, what is $\lim_{t \rightarrow +\infty} p(t)$?

decreases to $p=0$

$$\lim_{t \rightarrow \infty} p(t) = 0$$

- (e) Can a population of 900 ever increase to 1100?

No. Decrease to $p=0$