1.4 The Approximation Method of Euler

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + f(x_n, y_n) \end{cases}$$

1. Use Euler's method to approximate the solution to the given initial problem at the points $x = 0.1, 0.2, 0.3, 0.4, 0.5$, $h = 0.1$.

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -1$$

$f(x,y) = \frac{x}{y}$ $\quad x_0 = 0$, $y_0 = -1$

$$\begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + 0.1\frac{x_n}{y_n} \end{cases} \text{ OR (Given)}$$

$x_1 = 0 + 0.1 = 0.1$ $\quad y_1 = y_0 + 0.1\left(\frac{x_0}{y_0}\right) = -1 + 0.1\left(\frac{0}{-1}\right) = -1$

$x_2 = x_1 + 0.1 = 0.2$ $\quad y_2 = y_1 + 0.1\left(\frac{x_1}{y_1}\right) = -1 + 0.1\left(\frac{0.1}{-1}\right) = -1.01$

$x_3 = x_2 + 0.1 = 0.3$ $\quad y_3 = -1.01 + 0.1\left(\frac{0.2}{-1.01}\right) \approx -1.030$

$x_4 = x_3 + 0.1 = 0.4$ $\quad y_4 = -1.030 + 0.1\left(\frac{0.3}{-1.030}\right) \approx -1.059$

$x_5 = x_4 + 0.1 = 0.5$ $\quad y_5 = -1.059 + 0.1\left(\frac{0.4}{-1.059}\right) \approx -1.097$

2. Given the Initial Value Problem

$$y' = x - y, \quad y(0) = 0.$$

Actual : $y(1) = e^{-1} \approx 0.3679$

$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h(x_n - y_n) \end{cases} \quad x_0 = 0, \; y_0 = 0$$

The actual solution is $y = e^{-x} + x - 1$. Find a value of h for Euler's method such that $y(1)$ is approximated to within ± 0.01.

Calculator OR computer

h	$y(1)$
1	0
2^{-1}	0.25
2^{-2}	~~0.3436~~ 0.3164
2^{-3}	0.3436
2^{-4}	0.3561
2^{-5}	0.3621

$h = 1$, $x_1 = 1$, $y_1 = y_0 + 1(x_0 - y_0) = 0 + 1 \cdot 0 = 0$

$h = \frac{1}{2}$ $\quad x_1 = \frac{1}{2}$ $\quad y_1 = 0 + \frac{1}{2}(0 - 0) = 0$

$x_2 = 1$ $\quad y_2 = 0 + \frac{1}{2}\left(\frac{1}{2} - 0\right) = 0.25$

Stop when $h = 2^{-5}$ because

① $|\underbrace{0.3621}_{2^{-5}} - \underbrace{0.3561}_{2^{-4}}| < 0.01$ OR ② $|\underbrace{0.3621}_{2^{-5}} - \underbrace{0.3679}_{\text{Actual}}| < 0.01$

Also find, to within ± 0.05, the value of x_0 such that $y(x_0) = 0.2$.

Take $h = 0.1$ ⇐ Because (within ±0.05) ⇒ 0.1 is a best choice.

$x_1 = 0.1$ $\quad y_1 = 0 + 0.1(0 - 0) = 0$

$x_2 = 0.2$ $\quad y_2 = 0 + 0.1(0.1 - 0) = 0.01$

$x_3 = 0.3$ $\quad y_3 = 0.01 + 0.1(0.2 - 0.01) \approx 0.03$

$x_4 = 0.4$ $\quad y_4 = 0.03 + 0.1(0.3 - 0.03) \approx 0.06$

$x_5 = 0.5$ $\quad y_5 = 0.06 + 0.1(0.4 - 0.06) \approx 0.09$

$x_6 = 0.6$ $\quad y_6 = 0.09 + 0.1(0.5 - 0.09) \approx 0.13$

$x_7 = 0.7$ $\quad y_7 = 0.13 + 0.1(0.6 - 0.13) \approx 0.18 < 0.2$

$x_8 = 0.8$ $\quad y_8 = 0.18 + 0.1(0.7 - 0.18) \approx 0.23 > 0.2$

There is an x between 0.7 and 0.8 such that $y(x) = 0.2$

Take 0.75 since "within ±0.05"

Use Euler's method Calculator online
You can choose h=0.05
y(0.7)=0.1877, y(0.75)=0.2133
so we can choose any number between 0.7 and 0.75, for example x=0.74

2.2 Separable Equation

1. Solve the equation

(a) $\frac{dx}{dt} = \frac{t}{xe^{t+2x}}$ $= \frac{t}{xe^te^{2x}} = \frac{t}{e^t} \cdot \frac{1}{xe^{2x}}$

Separate: $xe^{2x}dx = te^{-t}dt$

~~I.B.P.~~ $\int xe^{2x}dx = \int te^{-t}dt$ Integrate

Simplify: $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} = -te^{-t} - e^{-t} + C$

Integration by parts: $udv = uv - vdu$

Left: $u = x$ $du = dx$

$dv = e^{2x}dx$ $v = \frac{1}{2}e^{2x}$

$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \Rightarrow \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$

Right: $u = t$ $du = dt$

$dv = e^{-t}dt$ $v = -e^{-t}$

$-te^{-t} - (-e^{-t})dt \Rightarrow -te^{-t} - e^{-t}$

2. Solve the initial value problem

(a) $\frac{1}{2}\frac{dy}{dx} = \sqrt{y+1}\cos x, \qquad y(\pi) = 0$

Separate: $\frac{1}{2}\frac{1}{\sqrt{y+1}}dy = \cos x\, dx$

Integrate: $\int \frac{1}{2}\frac{1}{\sqrt{y+1}}dy = \int \cos dx$

$(y+1)^{\frac{1}{2}} = \sin x + C$

Use $y(\pi) = 0 \Rightarrow x = \pi,\ y = 0$

$(0+1)^{\frac{1}{2}} = \sin\pi + C \Rightarrow C = 1$

$(y+1)^{\frac{1}{2}} = \sin x + 1$

$y = (\sin x + 1)^2 - 1$

(b) $x^2dx + 2ydy = 0, \qquad y(0) = 2$

Separate: $x^2dx = -2ydy$

Integrate: $\int x^2dx = \int -2ydy$

$\frac{1}{3}x^3 = -y^2 + C$

$x = 0,\ y = 2$

$0 = -4 + C \Rightarrow C = 4$

$-y^2 + 4 = \frac{1}{3}x^3$

$y^2 = 4 - \frac{1}{3}x^3 \Rightarrow y = \sqrt{4 - \frac{1}{3}x^3}$

$\cancel{y = -\sqrt{4 - \frac{1}{3}x^3}}$ Exclude b/c: $y(0) = 2$ not $y(0) = -2$