

10.2 Method of Separation of Variables

1. The heat flow problem or Heat equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \beta \frac{\partial^2 u(x,t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

PDE
(Dirichlet) Boundary Condition
Initial condition

$$u(x,t) = \Xi(x) \cdot T(t)$$

$$U_t = \Xi T' \quad \Rightarrow \quad \Xi T' = \beta \Xi'' T$$

$$U_{xx} = \Xi'' T \quad \downarrow$$

$$\frac{T'}{\beta T} = \frac{\Xi''}{\Xi} = \text{constant} = -\lambda$$

$$\Xi'' + \lambda \Xi = 0$$

$$T' + \lambda \beta T = 0$$

$$\text{B.C. } \Xi(0)T(t) = 0$$

$$\Xi(L)T(t) = 0$$

$$\Xi(0) = \Xi(L) = 0$$

Solve two ODEs

$$\begin{aligned} \textcircled{1} \quad \Xi'' + \lambda \Xi &= 0 \\ \Xi(0) &= \Xi(L) = 0 \end{aligned}$$

$$\textcircled{2} \quad T' + \lambda \beta T = 0$$

2. Determine all solutions to the boundary value problem

$$y'' + y = 0, \quad 0 < x < 2\pi; \quad y(0) = 0, \quad y(2\pi) = 1$$

$$y = e^{rt} \quad r^2 + 1 = 0 \quad \text{roots } \pm i, \quad y_1(t) = C_1 \cos t, \quad y_2(t) = C_2 \sin t$$

$$y(t) = C_1 \cos t + C_2 \sin t$$

$$y(0) = C_1 = 0, \quad C_1 = 0 \quad \Rightarrow \text{Impossible}$$

$$y(2\pi) = C_1 + 0 = 1, \quad C_1 = 1$$

There is No solution

3. Find the values of λ (eigenvalues) for which the boundary value problem has a nontrivial solution. Then determine the corresponding nontrivial solutions (eigenfunctions).

- (a) $y'' + \lambda y = 0, \quad 0 < x < \pi; \quad y(0) = 0, \quad y'(\pi) = 0$
- (b) $y'' + \lambda y = 0, \quad 0 < x < \pi; \quad y'(0) = 0, \quad y(\pi) = 0$
- (c) $y'' + \lambda y = 0, \quad 0 < x < 2\pi; \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$

$$y = e^{rt}, \quad r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda, \quad r = \pm\sqrt{\lambda}$$

Case 1: $\lambda < 0$, two real roots: $\sqrt{\lambda}, -\sqrt{\lambda}$, $y(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$
 $y'(x) = C_1 \sqrt{\lambda} e^{\sqrt{\lambda}x} + C_2 (-\sqrt{\lambda}) e^{-\sqrt{\lambda}x}$

Case 2: $\lambda = 0$, double roots: 0, $y(x) = C_1 + C_2 x$, $y'(x) = C_2$

Case 3: $\lambda > 0$, complex roots: $\pm\sqrt{\lambda} i$,

$$y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x), \quad y'(x) = -C_1(\sqrt{\lambda}) \sin(\sqrt{\lambda}x) + C_2(\sqrt{\lambda}) \cos(\sqrt{\lambda}x)$$

(a) Case 1: $y(0) = C_1 + C_2 = 0, \quad C_2 = -C_1 \quad \lambda < 0$

$$y'(\pi) = C_1 \sqrt{\lambda} e^{\sqrt{\lambda}\pi} + C_2 (-\sqrt{\lambda}) e^{-\sqrt{\lambda}\pi} = C_1 \sqrt{\lambda} (e^{\sqrt{\lambda}\pi} + e^{-\sqrt{\lambda}\pi}) = 0$$

If $C_1 \neq 0$, $e^{\sqrt{\lambda}\pi} + e^{-\sqrt{\lambda}\pi} > 0$, need $\sqrt{\lambda} = 0$, or $\lambda = 0$ impossible

Case 2: $\lambda = 0, \quad y(0) = C_1 = 0, \quad y'(\pi) = C_2 = 0$. Trivial

Case 3: $\lambda > 0$

$$y(0) = C_1 + 0 = 0 \quad C_1 = 0$$

$$y'(\pi) = C_2(\sqrt{\lambda}) \cos(\sqrt{\lambda}\pi) = 0 \quad \text{want } C_2 \neq 0$$

$$\cos(\sqrt{\lambda}\pi) = 0 \quad \sqrt{\lambda}\pi = n\pi - \frac{\pi}{2}, \quad n=1, 2, 3, \dots$$

$$\sqrt{\lambda} = n - \frac{1}{2}, \quad \lambda_n = (n - \frac{1}{2})^2 \leftarrow \text{eigenvalues}$$

$$y_n(x) = C_n \sin((n - \frac{1}{2})x) \quad \text{OR eigenfunctions } \sin((n - \frac{1}{2})x)$$

$$(b) \text{ Case 1: } \lambda < 0 \quad y(0) = C_1(\sqrt{-\lambda}) + C_2(-\sqrt{-\lambda}) = C_1\sqrt{-\lambda} - C_2\sqrt{-\lambda} = 0$$

$$C_1 = C_2$$

$$y(\pi) = C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} = C_1 \left(e^{\frac{\sqrt{-\lambda}\pi}{2}} + e^{-\frac{\sqrt{-\lambda}\pi}{2}} \right) = 0$$

$$C_1 = 0 = C_2 \quad \text{Trivial}$$

$$\text{Case 2: } y'(0) = C_2 = 0, \quad y(\pi) = C_1 = 0 \quad \text{Trivial}$$

$$\text{Case 3: } \lambda > 0 \quad y'(0) = C_2(\sqrt{\lambda}) = 0 \quad C_2 = 0$$

$$y(\pi) = C_1 \cos(\sqrt{\lambda}\pi) = 0 \quad \sqrt{\lambda}\pi = n\pi - \frac{1}{2}\pi \quad n=1, 2, 3, \dots$$

$$\lambda_n = (n - \frac{1}{2})^2 \quad \text{eigenvalues}$$

$$y_n = C_n \cos((n - \frac{1}{2})x) \quad \text{eigenfunctions}$$

$$(c) \text{ Case 1: } \lambda < 0 \quad y(0) = C_1 + C_2 = C_1 e^{2\pi\sqrt{-\lambda}} + C_2 e^{-2\pi\sqrt{-\lambda}} = y(2\pi)$$

$$y'(0) = C_1(\sqrt{-\lambda}) + C_2(-\sqrt{-\lambda}) = C_1(\sqrt{-\lambda}) e^{2\pi\sqrt{-\lambda}} + C_2(-\sqrt{-\lambda}) e^{-2\pi\sqrt{-\lambda}} = y'(2\pi)$$

$$\begin{cases} C_1 + C_2 = C_1 e^{2\pi\sqrt{-\lambda}} + C_2 e^{-2\pi\sqrt{-\lambda}} & ① \\ C_1 - C_2 = C_1 e^{2\pi\sqrt{-\lambda}} - C_2 e^{-2\pi\sqrt{-\lambda}} & ② \end{cases} \quad ① + ② \quad 2C_1 = 2C_1 e^{2\pi\sqrt{-\lambda}} \quad C_1(e^{2\pi\sqrt{-\lambda}} - 1) = 0, \quad C_1 = 0$$

$$① \quad C_2 = C_1 e^{-2\pi\sqrt{-\lambda}} \Rightarrow C_2 \left(e^{-2\pi\sqrt{-\lambda}} - 1 \right) = 0 \quad C_2 = 0 \quad \text{Trivial}$$

$$\text{Case 2: } \lambda = 0 \quad y(0) = C_1 = C_1 + 2\pi C_2 = y(2\pi) \quad C_2 = 0$$

$$y'(0) = C_2 = y'(2\pi) \quad \checkmark \quad C_1 \text{ can be any constant.}$$

$$\lambda = 0, \quad y(x) = C_0$$

$$\text{Case 3: } \lambda > 0 \quad y(0) = C_1 = C_1 \cos(2\pi\sqrt{\lambda}) + C_2 \sin(2\pi\sqrt{\lambda}) = y(2\pi)$$

$$y'(0) = C_2 \cancel{\sqrt{\lambda}} = -C_1 \cancel{\sqrt{\lambda}} \sin(2\pi\sqrt{\lambda}) + C_2 \cancel{\sqrt{\lambda}} \cos(2\pi\sqrt{\lambda}) = y'(2\pi)$$

$$\begin{cases} C_1 = C_1 \cos(2\pi\sqrt{\lambda}) + C_2 \sin(2\pi\sqrt{\lambda}) \\ C_2 = -C_1 \sin(2\pi\sqrt{\lambda}) + C_2 \cos(2\pi\sqrt{\lambda}) \end{cases}$$

$$C_1(\cos(2\pi\bar{\lambda}) - 1) + C_2 \sin(2\pi\bar{\lambda}) = 0 \quad ①$$

$$-C_1 \sin(2\pi\bar{\lambda}) + C_2(\cos(2\pi\bar{\lambda}) - 1) = 0 \quad ②$$

$$① \times \sin(2\pi\bar{\lambda}) + ② \times (\cos(2\pi\bar{\lambda}) - 1) \quad \text{cancel } C_1$$

$$C_2 \sin^2(2\pi\bar{\lambda}) + C_2(\cos(2\pi\bar{\lambda}) - 1)^2 = 0$$

$$C_2 \left(\sin^2(2\pi\bar{\lambda}) + \overset{2}{\cos(2\pi\bar{\lambda})} \overset{=1}{\rightarrow} - 2\cos(2\pi\bar{\lambda}) + 1 \right) = 0$$

$$C_2(2 - 2\cos(2\pi\bar{\lambda})) = 0$$

$$① \times (\cos(2\pi\bar{\lambda}) - 1) - ② \times \sin(2\pi\bar{\lambda}) \quad \text{cancel } C_2$$

$$C_1(\cos(2\pi\bar{\lambda}) - 1)^2 + C_1 \sin^2(2\pi\bar{\lambda}) = 0$$

$$C_1(\cos^2(2\pi\bar{\lambda}) - 2\cos(2\pi\bar{\lambda}) + 1 + \sin^2(2\pi\bar{\lambda})) = 0$$

$$C_1(2 - 2\cos(2\pi\bar{\lambda})) = 0$$

$$\rightarrow \text{want : } 2 - 2\cos(2\pi\bar{\lambda}) = 0$$

$$\cos(2\pi\bar{\lambda}) = 1$$

$$2\pi\bar{\lambda} = 2n\pi, \quad n=1, 2, 3, \dots$$

$$\lambda_n = n^2, \quad \text{eigenvalues}$$

$$y_{n(x)} = a_n \cos(nx) + b_n \sin(nx) \quad \text{eigenfunctions.}$$

4. Solve the heat flow problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \beta \frac{\partial^2 u(x,t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

Heat Equation has Dirichlet boundary conditions

with $\beta = 3$, $L = \pi$, and the given function $f(x) = \sin 3x + 5 \sin 7x - 2 \sin 13x$.

Formula sheet: PDF 1(a), $\beta = 3, L = \pi$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3\left(\frac{n\pi}{\pi}\right)^2 t} \sin\left(\frac{n\pi}{\pi}x\right) = \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin(nx)$$

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin(nx) = f(x)$$

$$\begin{array}{cccc} \underline{\sin 3x} & \underline{\sin 7x} & \underline{\sin 13x} & \\ \hline n=3 & n=7 & n=13 & \\ b_3=1 & b_7=5 & b_{13}=-2 & \text{other } b_n=0 \end{array}$$

$$\begin{aligned} u(x,t) &= 1 e^{-3(3)^2 t} \sin(3x) + 5 e^{-3(7)^2 t} \sin(7x) - 2 e^{-3(13)^2 t} \sin(13x) \\ &= e^{-27t} \sin(3x) + 5e^{-147t} \sin(7x) - 2e^{-507t} \sin(13x) \end{aligned}$$

5. Solve the vibrating string problem

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & 0 \leq x \leq L \\ \frac{\partial u}{\partial t}(x,0) = g(x), & 0 \leq x \leq L \end{cases}$$

with $\alpha = 3$, $L = \pi$, and the given initial functions

$$f(x) = \sin x - 2 \sin 2x + \sin 3x, \quad g(x) = 6 \sin 3x - 7 \sin 5x.$$

Formula 2 : $\alpha = 3$, $L = \pi$

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi \cdot 3}{\pi} t\right) + b_n \sin\left(\frac{n\pi \cdot 3}{\pi} t\right)] \sin\left(\frac{n\pi x}{\pi}\right) \\ &= \sum_{n=1}^{\infty} [a_n \cos(3nt) + b_n \sin(3nt)] \sin(nx) \end{aligned}$$

$$u(x,0) = \sum_{n=1}^{\infty} [a_n] \sin(nx) = f(x) = \underbrace{\sin(1x)}_{n=1} - 2\underbrace{\sin(2x)}_{n=2} + \underbrace{\sin(3x)}_{n=3}$$

$$a_1 = 1 \quad a_2 = -2 \quad a_3 = 1$$

other $a_n = 0$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} [-a_n \sin(3nt)(3n) + b_n \cos(3nt)(3n)] \sin(nx)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} [b_n(3n)] \sin(nx) = g(x) = \underbrace{6 \sin(3x)}_{n=3} - \underbrace{7 \sin(5x)}_{n=5}$$

$$b_3(3 \cdot 3) = 6 \quad b_5(3 \cdot 5) = -7$$

$$b_3 = \frac{2}{3} \quad b_5 = -\frac{7}{15}$$

$$u(x,t) = \cos(3t)\sin(x) - 2\cos(6t)\sin(2x)$$

$n=1 \qquad \qquad n=2 \qquad \qquad \text{other } b_n = 0$

$$+ \sum_{n=3}^{\infty} [\cos(9t) + \frac{2}{3} \sin(9t)] \sin(nx) - \frac{7}{15} \sin(15t) \sin(5x)$$

$n=3 \qquad \qquad \qquad n=5$