

## 10.3 Fourier Series &amp; 10.4 Fourier Cosine and Sine Series

1. Compute the Fourier series for function  $f(x) = |x|$ ,  $-\pi < x < \pi$ . even function,  $L = \pi$

$$|x| \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \stackrel{\text{even}}{=} \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \frac{x^2}{2} \Big|_0^{\pi} = \frac{2}{\pi} \frac{\pi^2}{2} = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx \stackrel{\text{even}}{=} \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx$$

$$= \frac{2}{\pi} \left[ x \frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_{x=0}^{x=\pi} \quad \sin(n\pi) = \sin(0) = 0$$

$$= \frac{2}{\pi} \left( \frac{\cos(n\pi)}{n^2} - \frac{\cos(0)}{n^2} \right) = \frac{2}{n^2\pi} [(-1)^n - 1]$$

$$= \begin{cases} \frac{2}{n^2\pi} (-2) = \frac{-4}{n^2\pi} & n = \text{odd} \\ 0 & n = \text{even} \end{cases} \Rightarrow \underline{\text{Not required}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \underbrace{\sin(nx) dx}_{\text{even} \cdot \text{odd}} = 0$$

$$|x| \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} [(-1)^n - 1] \cos(nx)$$

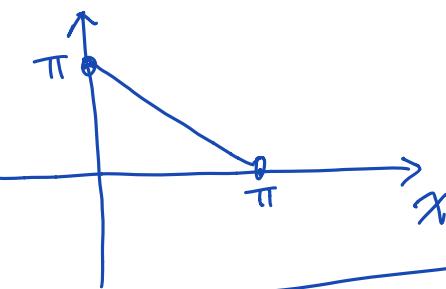
$$= \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi} \frac{1}{(2k-1)^2} \cos((2k-1)x) \quad n = \text{odd} = 2k-1$$

$\uparrow$   
not required

even

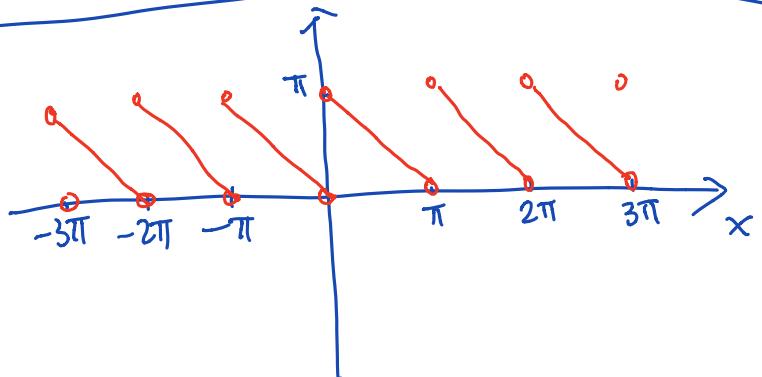
2. Determine the  $\pi$ -periodic extension  $\tilde{f}$ , the odd  $2\pi$ -periodic extension  $f_o$  and the ~~odd~~ even  $2\pi$ -periodic extension  $f_e$  for  $f(x) = \pi - x$ ,  $0 < x < \pi$ .

$$f(x) = \pi - x$$



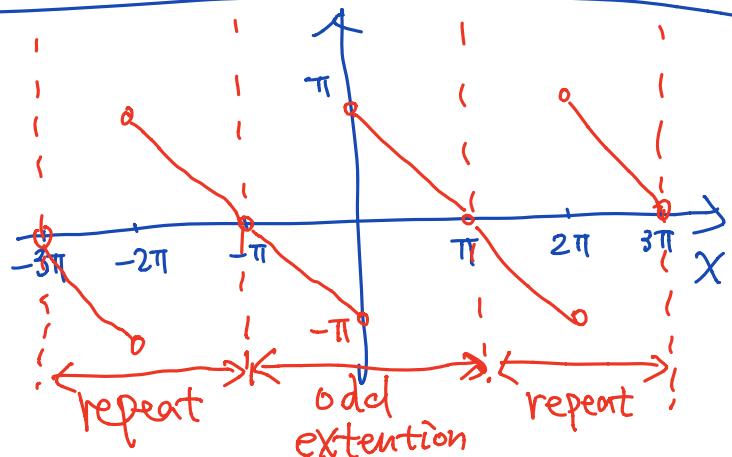
$$\tilde{f}(x) = \pi - x, 0 < x < \pi$$

$$\tilde{f}(x+\pi) = \tilde{f}(x) \text{ } \pi\text{-periodic}$$



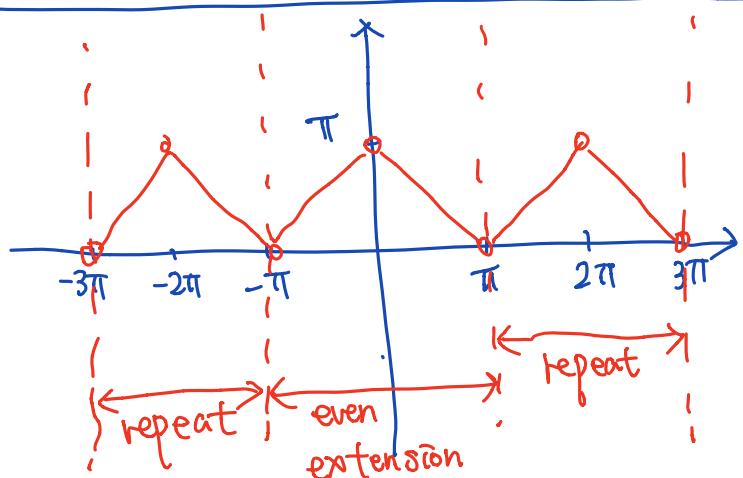
$$f_o(x) = \begin{cases} \pi - x & 0 < x < \pi \\ -(\pi - (-x)) = -\pi - x & -\pi < x < 0 \end{cases}$$

$$\text{with } f_o(x+2\pi) = f_o(x)$$



$$f_e(x) = \begin{cases} \pi - x & 0 < x < \pi \\ \pi - (-x) = \pi + x & -\pi < x < 0 \end{cases}$$

$$\text{with } f_e(x+2\pi) = f_e(x)$$



3. Determine the Fourier sine series and cosine series for  $f(x) = \pi - x$ ,  $0 < x < \pi$ .

Fourier Sine :  $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ ,  $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$\pi - x \sim \sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) \quad \leftarrow$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(nx) dx = \frac{2}{\pi} \left( \int_0^{\pi} \pi \sin(nx) dx - \int_0^{\pi} x \sin(nx) dx \right) \\ &= \frac{2}{\pi} \left( \pi \frac{-\cos(nx)}{n} \Big|_0^{\pi} - \left[ -x \frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi} \right) \\ &= \frac{2}{\pi} \left[ \cancel{\pi \frac{-\cos(n\pi)}{n}} - \pi \frac{-1}{n} - \cancel{\left( -\pi \frac{\cos(n\pi)}{n} - 0 \right)} \right] = \frac{2}{\pi} \cdot \frac{\pi}{n} = \frac{2}{n} \end{aligned}$$

Fourier cosine :  $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ ,  $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$$\pi - x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left[ \pi x - \frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \left( \pi^2 - \frac{\pi^2}{2} \right) = \pi$$

$$\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [(-1)^n + 1] \cos(nx)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx) dx$$

$$= \frac{2}{\pi} \left( \int_0^{\pi} \pi \cos(nx) dx - \int_0^{\pi} x \cos(nx) dx \right)$$

$$= \frac{2}{\pi} \left[ \pi \frac{\sin(nx)}{n} \Big|_0^{\pi} - \left( x \frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi} \left[ - \left( \frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right) \right] = \frac{2}{\pi n^2} [ -(-1)^n + 1 ]$$

Formula you may need (but use WolframAlpha  
is easier).

$$\int x \sin(nx) dx \stackrel{\text{IBP}}{=} -x \frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$$

$$\int x \cos(nx) dx \stackrel{\text{IBP}}{=} x \frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2} + C$$

$$\int e^{ax} \cos(bx) dx = \frac{1}{a^2 + b^2} [be^{ax} \sin(bx) + ae^{ax} \cos(bx)] + C$$

$$\int e^{ax} \sin(bx) dx = \frac{1}{a^2 + b^2} [-be^{ax} \cos(bx) + ae^{ax} \sin(bx)] + C$$