

The Heat and Wave Equations

1. Find a formal solution to the given initial-boundary value problem.

$$\begin{cases} \frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \quad t > 0 \\ \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, & t > 0 \\ u(x, 0) = x, & 0 < x < \pi \end{cases} \quad \beta = 3, L = \pi$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-3n^2 t} \cos(nx)$$

$$u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = x$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \frac{\pi^2}{2} = \pi$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx = \frac{2}{\pi} \left[x \frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \frac{1}{n^2} [\cos(n\pi) - \cos(0)] = \frac{2}{\pi n^2} [(-1)^n - 1] \end{aligned}$$

$$\underline{\underline{\text{Sol}}} : \quad u(x, t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] e^{-3n^2 t} \cos(nx)$$

2. Find a formal solution to the wave equation governed by the given initial-boundary value problem.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, & 0 < x < 1, \quad t > 0 \\ u(0, t) = u(1, t) = 0, & t > 0 \\ u(x, 0) = x(1-x), & 0 < x < 1 \\ \frac{\partial u}{\partial t}(x, 0) = \sin 7\pi x, & 0 < x < 1 \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)] \sin(n\pi x)$$

$$u(x, 0) = \sum_{n=1}^{\infty} [a_n] \sin(n\pi x) = x(1-x)$$

$$a_n = \frac{2}{1} \int_0^1 x(1-x) \sin(n\pi x) dx$$

$$= 2 \left[\frac{-x(1-x)\cos(n\pi x)}{n\pi} + (1-2x) \frac{\sin(n\pi x)}{(n\pi)^2} - \frac{2\cos(n\pi x)}{(n\pi)^3} \right] \Big|_0^1$$

$$= 2 \left[0 - 0 - \left(\frac{2\cos(0)}{n^3\pi^3} - \frac{2\cos(0)}{n^3\pi^3} \right) \right] = \frac{4}{n^3\pi^3} [(-1)^n + 1]$$

$$\begin{aligned} & x(1-x) \xrightarrow{+} \sin(n\pi x) \\ & 1-2x \xrightarrow{-} \frac{\cos(n\pi x)}{n\pi} \\ & -2 \xrightarrow{+} -\frac{\sin(n\pi x)}{(n\pi)^2} \\ & 0 \xrightarrow{+} \frac{\cos(n\pi x)}{(n\pi)^3}. \end{aligned}$$

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} [0 + b_n(n\pi) \cos(0)] \sin(n\pi x) = \sum_{n=1}^{\infty} b_n(n\pi) \sin(n\pi x)$$

$$b_7(7\pi) = 1, \quad b_7 = \frac{1}{7\pi}, \quad \text{other } b_n = 0 = \sin 7\pi x \quad n=7$$

$$u(x, t) = \frac{1}{7\pi} \sin(7\pi t) \sin(7\pi x) + \sum_{n=1}^{\infty} \frac{\frac{4}{n^3\pi^3} [(-1)^n + 1]}{a_n} \cos(n\pi t) \sin(n\pi x)$$