

3.2 Compartmental Analysis

1. A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3 L/min. If the concentration of salt in the brine entering the tank is 0.2 kg/L, determine the mass of salt in the tank after t min. When will the concentration of the salt in the tank reach 0.1 kg/L?

$X(t)$ — mass of salt at time t , $X(0) = 0$ (kg) pure water

$$\text{Input rate} = 4 \text{ L/min} \cdot 0.2 \text{ kg/L} = 0.8 \text{ kg/min}$$

$$\text{Output rate} = 3 \text{ L/min} \cdot \text{concentration}$$

$$\text{concentration} = \frac{\text{mass}}{\text{volume}} = \frac{X(t)}{100 + (4-3)t} = \frac{X}{100+t}$$

$$\text{volume} = 100 + t \text{ (in-out)} = 100 + (4-3)t = V(t)$$

The IVP:

$$\begin{cases} \frac{dx}{dt} = 0.8 - \frac{3x}{100+t} & (\text{in-out}) \\ X(0) = 0 \end{cases}$$

Solve: $\frac{dx}{dt} + \frac{3}{100+t} x = 0.8$ Linear Equation

$$3.4 \text{ Newtonian Mechanics} \quad \text{Concentration} = \frac{X(t)}{V(t)} = 0.2 - \frac{2 \cdot 10^7}{(100+t)^4} = 0.1, \text{ solve for } t$$

1. An object of mass 8 kg is given an upward initial velocity of 20 m/sec and then allowed to fall under influence of gravity. Assume that the force in newtons due to air resistance is $-16v$, where v is the velocity of the object in m/sec. Determine the equation of motion of the object. If the object is initially 100 m above the ground, determine when the object will strike the ground.

$x(0) = 0$ $v_0 = -20 \text{ m/sec}$

Newton's Second Law $ma = F_{\text{total}}$

$$m \frac{dv}{dt} = mg + F = 9.81m - 16v \Rightarrow \frac{dv}{dt} = 9.81 - 2v$$

IVP $\begin{cases} \frac{dv}{dt} = 9.81 - 2v \\ v(0) = -20 \end{cases}$

$$x(t) = 100 \text{ solve for } t$$

$$e^{-2t} \text{ is small } \approx 0$$

$$x(t) = 4.905t - 12.4525 = 100$$

$$t \approx 22.9$$

Forces

$$\uparrow F = -16v$$

$$\downarrow mg$$

Solve $\frac{dv}{dt} + 2v = 9.81$ Linear

$$u = e^{\int 2dt} = e^{2t}$$

$$(uv) = \int 9.81e^{2t} dt = 4.905e^{2t} + C$$

$$v(t) = 4.905 + Ce^{-2t}$$

$$v(0) = 4.905 + C = -20 \quad C = -24.905$$

$$v(t) = 4.905 - 24.905e^{-2t}$$

Find distance: $\begin{cases} x'(t) = v(t) \\ x(0) = 0 \end{cases}$

$$x(t) = \int v(t) dt = \int 4.905 + Ce^{-2t} dt$$

$$= \int (4.905 - 24.905e^{-2t}) dt$$

$$= 4.905t + 12.4525e^{-2t} + C$$

$$x(0) = 12.4525 + C = 0 \quad C = -12.4525$$

$$x(t) = 4.905t + 12.4525e^{-2t} - 12.4525$$