

4.4 Nonhomogeneous Equations: the Method of Undetermined Coefficients  
 4.5 The Superposition Principle and Undetermined Coefficients Revisited

1. Find a general solution to the differential equations  $y'' + 4y = \sin t - \cos t$ .

Homogeneous:  $y'' + 4y = 0$       Guess:  $y_h = e^{rt}$

 $r^2 + 4 = 0, \rightarrow r = \pm 2i$ 

$f(t) = \sin t - \cos t = -I e^{ot} \cos t + I e^{ot} \sin t$

$0 \pm 2i$  is not a root

$P_m(t) = -1, Q_n(t) = 1, \text{ degree } = 0$

$y_p = A \cos t + B \sin t$

$y'_p = -A \sin t + B \cos t$

$y''_p = -A \cos t - B \sin t$

$y'' + 4y = -A \cos t - B \sin t + 4(A \cos t + B \sin t) = \sin t - \cos t$

$A = -\frac{1}{3}, B = \frac{1}{3}$

$y_p = -\frac{1}{3} \cos t + \frac{1}{3} \sin t$

General sol:  $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + y_p$

2. Find the solution to the initial value problem:  $y'' = 6t, y(0) = 3, y'(0) = -1$

Homo:  $y'' = 0$       Guess:  $y_h = e^{rt}$

 $r^2 = 0, r = 0$  double
  $y_1 = e^{ot}, y_2 = te^{ot}$ 

$y_h = C_1 + C_2 t = C_1 + C_2 t$

$y_p = t(At + B) = At^2 + Bt$

$y'_p = 3At^2 + 2Bt$

$f(t) = 6t = 6te^{ot}$

$0$  is a double root,  $S = 2$

$P_m(t) = 6t$  degree = 1

$y'' = 6At + 2B = 6t$

$A = 1, B = 0$

$y_p(t) = t^3, y_g = C_1 + C_2 t + t^3$

$y_p \Rightarrow -\frac{1}{3} \cos t + \frac{1}{3} \sin t$

$y(0) = C_1 = 3 \Rightarrow C_1 = 3$

$y'(0) = C_2 + 3t^2. y'(0) = C_2 = -1, C_2 = -1$

Answer:  $y(t) = 3 - t + t^3$

3. Determine the form of a particular solution for the differential equation (do not evaluate coefficients).

$y''_p = 6At + 2B$

(a)  $y'' + 2y' - y = 10$

Homo:  $y'' + 2y' - y = 0$

 $r^2 + 2r - 1 = 0$ 

roots:  $-1 \pm \sqrt{2}$

$y_p(t) = A$

$f(t) = 10 = 10e^{ot}$

$0$  is not a root

(d)  $y'' - 2y' + y = 7e^t \cos t$

Homo:  $y'' - 2y' + y = 0$

 $r^2 - 2r + 1 = 0$ 

double roots = 1

$y_p(t) = Ae^t \cos t + Be^t \sin t$

$f = 7e^t \cos t$

$1+1i$  is not a root

(b)  $y'' + 4y = 8 \sin 2t$

Homo:  $y'' + 4y = 0$

 $r^2 + 4 = 0, \text{ roots } \pm 2i$ 

$y_p(t) = A \cos t \cdot t + B \sin t \cdot t$

$f(t) = 8 \sin 2t$

$= 8e^{ot} \sin 2t$

$0+2i$  is a root

(c)  $x'' - 4x' + 4x = te^{2t}$

Homo:  $x'' - 4x' + 4x = 0$

 $r^2 - 4r + 4 = 0$ 

double roots = 2

$y_p(t) = t^2 (At + B) e^{2t}$

$f(t) = te^{2t}$

2 is double,  $S = 2$

$\uparrow$   
degree = 1

(e)  $y'' - 2y' - 3y = 3t^2 - 5$

Homo:  $y'' - 2y' - 3y = 0$

 $r^2 - 2r - 3 = 0$ 

roots:  $3, -1$

$y_p(t) = At^2 + Bt + C$

$f = 3t^2 - 5$

degree = 2,  
 $0$  is not a root

(f)  $y'' + y' - 12y = e^t + e^{2t} - 1$

Homo:  $y'' + y' - 12y = 0$

 $r^2 + r - 12 = 0$ 

$(r-3)(r+4) = 0$

roots:  $3, -4$

$y_p(t) = Ae^t + Be^{2t} + C$

$f(t) = e^t + e^{2t} - 1$

$e^{ot}, 0$  not root  
degree = 0

1  $\uparrow$  not a root  
degree = 0

2 not root  
degree = 0