

## 4.7 Variable-Coefficient Equations

1. Find a general solution to the Cauchy-Euler equation for  $t > 0$ .

$$(a) t^2 \frac{d^2y}{dt^2} + 2t \frac{dy}{dt} - 6y = 0 \quad \text{Guess } y = t^r$$

Characteristic Eqn:

$$r^2 + (2-1)r - 6 = 0$$

$$r^2 + r - 6 = (r+3)(r-2) = 0$$

$$\text{roots: } -3, 2, \quad y_1 = t^{-3}, \quad y_2 = t^2$$

$$y_h(t) = C_1 t^{-3} + C_2 t^2$$

$$(b) t^2 \frac{d^2y}{dt^2} + 5t \frac{dy}{dt} + 4y = 0$$

$$\text{Guess } y = t^r, \quad r^2 + (5-1)r + 4 = 0$$

$$r^2 + 4r + 4 = (r+2)^2 = 0 \quad \text{double root} = -2$$

$$y_1 = t^{-2}, \quad y_2 = t^{-2} \ln t$$

$$y_h(t) = C_1 t^{-2} + C_2 t^{-2} \ln t$$

2. Find a general solution to the Cauchy-Euler equation for  $t < 0$ :  $t^2 y''(t) + 3ty'(t) + 5y(t) = 0$ .

$$\text{Guess } y = t^r, \quad r^2 + (3-1)r + 5 = 0$$

$$r^2 + 2r + 5 = 0 \quad r = \frac{-2 \pm \sqrt{4-45}}{2} = \frac{-2 \pm 4i}{2}$$

$$\text{roots: } -1 \pm 2i, \quad \alpha = -1, \quad \beta = 2$$

$$y_1 = t^{-1} \cos(2 \ln t), \quad y_2 = t^{-1} \sin(2 \ln t)$$

3. Find a general solution to the equation for  $t > 0$ :  $t^2 y'' - 4ty' + 6y = t^3 + 1$ .

$$\text{Homogeneous: Guess } y = t^r, \quad r^2 + (-4-1)r + 6 = r^2 - 5r + 6 = 0$$

$$r^2 - 5r + 6 = (r-2)(r-3) = 0, \quad \text{roots: } 2, 3$$

$$y_1 = t^2, \quad y_2 = t^3, \quad y_h(t) = C_1 t^2 + C_2 t^3$$

$$\text{Rewrite: } y'' - \frac{4}{t} y' + \frac{6}{t^2} y = t + \frac{1}{t^2}$$

$$y_p(t) = v_1 y_1 + v_2 y_2 = t^2 v_1 + t^3 v_2$$

$$\begin{cases} v_1' t^2 + v_2' t^3 = 0 \\ v_1' (2t) + v_2' (3t^2) = t + \frac{1}{t^2} \end{cases} \quad \begin{array}{l} (1) \quad v_1' = -t v_2' \\ (2) \end{array}$$

4. Find a second linearly independent solution using reduction of order:  
 $t^2 y'' + 6ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^{-2}$

$$\text{Guess: } y_2(t) = t^{-2} v(t)$$

$$y_2' = -2t^{-3} v + t^{-2} v', \quad y_2'' = 6t^{-4} v + 2t^{-3} v' - 2t^{-3} v' + t^{-2} v'' = 6t^{-4} v - 4t^{-3} v' + t^{-2} v''$$

$$t^2 y_2'' + 6ty_2' + 6y_2 = 6t^{-2} v - 4t^{-1} v' + v'' + (-12t^{-3} v + 6t^{-2} v') + 6t^{-3} v = 0$$

$$v(6t^{-2} - 12t^{-3} + 6t^{-2}) + v'(-4t^{-1} + 6t^{-1}) + v'' = 0$$

$$\begin{aligned} y_2(t) &= t^{-2}(-t^{-1}) \\ &= -t^{-3} \end{aligned}$$

Answer!

If  $t < 0$ , replace  $t$  by  $-t$

$$y_1 = -t^{-1} \cos(2 \ln(-t))$$

$$y_2 = -t^{-1} \sin(2 \ln(-t))$$

$$y_h(t) = C_1 t^{-1} \cos(2 \ln(-t)) + C_2 t^{-1} \sin(2 \ln(-t))$$

$$(2): (-t v_2') + (2t) + 3t^2 v_2' = t + t^{-2}$$

$$-2t^2 v_2' + 3t^3 v_2' = t + t^{-2}$$

$$t^2 v_2' = t + t^{-2}, \quad v_2' = t^{-1} + t^{-4}$$

$$v_1' = -t v_2' = -1 - t^{-3}$$

$$v_1(t) = \int -1 - t^{-3} dt = -t + \frac{1}{2} t^{-2}$$

$$v_2(t) = \int t^{-1} + t^{-4} dt = \ln t - \frac{1}{3} t^{-3}$$

$$y_p(t) = t^2 \left( -t + \frac{1}{2} t^{-2} \right) + t^3 \left( \ln t - \frac{1}{3} t^{-3} \right) = -t^3 + \frac{1}{2} t^2 + t^3 \ln t - \frac{1}{6} = \frac{1}{6}$$

$$y_g(t) = C_1 t^2 + C_2 t^3 - t^3 + t^3 \ln t + \frac{1}{6}$$

$$y_p(t) = C_1 t^2 + C_2 t^3 + t^3 \ln t + \frac{1}{6}$$

$$v'' + 2t^{-1} v' = 0$$

$$\text{Let } w(t) = v'(t)$$

$$w' + 2t^{-1} w = 0$$

$$w' + \frac{2}{t} w = 0 \quad \frac{dw}{dt} = -\frac{2}{t} w$$

$$\frac{dw}{w} = -\frac{2}{t} dt \Rightarrow \ln w = -2 \ln t$$

$$w(t) = t^{-2}, \quad v'(t) = t^{-2}$$

$$v(t) = \int t^{-2} dt = -\frac{1}{3} t^{-3} - t^{-1}$$