

7.5 IVP by Laplace Transform

$$\mathcal{L}\{y''\}(s) = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\}(s) = s Y(s) - y(0)$$

Method of Laplace transforms:

- Take Laplace transform of both sides of the equation.
- Use the properties of Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and solve the equation.
- Determine the inverse Laplace transform of the solution.

$$1. y'' - 4y' + 5y = 4e^{3t}, \quad y(0) = 2, \quad y'(0) = 7$$

$$\textcircled{1} \quad \mathcal{L}\{y''\}(s) - 4\mathcal{L}\{y'\}(s) + 5\mathcal{L}\{y\} = 4\mathcal{L}\{e^{3t}\}$$

$$(s^2 Y - 2s - 7) - 4(sY - 2) + 5Y = \frac{4}{s-3}$$

$$\textcircled{2} \quad \mathcal{L}\{s^2 - 4s + 5\} Y(s) - 2s - 7 + 8 = \frac{4}{s-3}$$

$$[(s-2)^2 + 1] Y(s) = \frac{4}{s-3} + 2s - 1 = \frac{2s^2 - 7s + 7}{s-3}$$

$$Y(s) = \frac{2s^2 - 7s + 7}{(s-3)[(s-2)^2 + 1]}$$

$$\textcircled{3} \quad Y(s) = \frac{A}{s-3} + \frac{B(s-2) + C(1)}{(s-2)^2 + 1}$$

$$2s^2 - 7s + 7 = A[(s-2)^2 + 1] + B(s-2)(s-3) + C(s-3)$$

$$2. y'' - 7y' + 10y = 9\cos t + 7\sin t, \quad y(0) = 5, \quad y'(0) = -4$$

$$\textcircled{1} \quad \mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 9\mathcal{L}\{\cos t\} + 7\mathcal{L}\{\sin t\}$$

$$(s^2 Y - s\cancel{+10}) - 7(sY - 5) + 10Y$$

$$= \frac{9s}{s^2 + 1} + \frac{7}{s^2 + 1}$$

$$\textcircled{2} \quad \cancel{s^2 + 12s}[s^2 - 7s + 10] Y(s) - 5s + 4 + 35 = \frac{9s + 7}{s^2 + 1}$$

$$(s-2)(s-5) Y(s) = \frac{9s + 7}{s^2 + 1} + 5s - 39 =$$

$$= \frac{9s^3 - 39s^2 + 14s - 32}{s^2 + 1}$$

$$Y(s) = \frac{5s^3 - 39s^2 + 14s - 32}{(s-2)(s-5)(s^2 + 1)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{Cs + D}{s^2 + 1}$$

$$\textcircled{3} \quad 5s^3 - 39s^2 + 14s - 32 = A(s-5)(s^2 + 1)$$

$$B(s-2)(s^2 + 1)$$

$$Cs(s-2)(s-5)$$

$$Ds(s-2)(s-5)$$

$$S=3, \quad 18 - 21 + 7 = 2A + 0 + 0$$

$$4 = 2A \Rightarrow \boxed{A=2}$$

$$S=2, \quad 8 - 14 + 7 = 2[1] + 0 + C(-1)$$

$$1 = 2 - C \Rightarrow \boxed{C=1}$$

$$S=0, \quad 7 = 2[5] + B(-2)(-3) + 1(-3)$$

$$7 = 10 + 6B - 3 \Rightarrow B = 0$$

$$Y(s) = \frac{2}{s-3} + \frac{1}{(s-2)^2 + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2 + 1}\right\} t$$

$$\boxed{y(t) = 2e^{3t} + e^{2t} \sin t}$$

$$S=2, \quad 40 - 156 + 28 - 32 = A(-3)(5)$$

$$-120 = -15A \Rightarrow \boxed{A=8}$$

$$S=5, \quad \cancel{5s^2 + 12s} - 39 \cdot 25 + 14 \cdot 5 - 32 = B(3)(12)$$

$$25(25 - 39) + 14 \cdot 5 - 32 = 3 \cdot 26 B$$

$$25(-14) + 14 \cdot 5 - 32 = 3 \cdot 26 B$$

$$14(-20) - 32 = -312 = 3 \cdot 26 B$$

$$\boxed{B=-4}$$

$$S=0, \quad -32 = 8(-5) + B(-4)(-2) + D(-2)(-1)$$

$$-32 = -40 + 8 + 10D \Rightarrow \boxed{D=0}$$

$$S=1, \quad 5 - 39 + 14 - 32 = 8(-4)(1)$$

$$+ (-4)(-1)(2) + C(-1)(-4)$$

$$-52 = -64 + 8 + 4C \Rightarrow \boxed{C=1}$$

$$Y(s) = 8\frac{1}{s-2} + (-4)\frac{1}{s-5} + \frac{1}{s^2 + 1}$$

$$\boxed{y(t) = 8e^{2t} - 4e^{5t} + \cos t}$$