MATHEMATICS 220: EXAM I
University of Illinois at Chicago (Professor Nicholls)
October 6, 2017

Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Solve the initial value problem
$$\frac{dy}{dx} = \frac{x^2+1}{x(\cos(y)+2e^y)}, \quad y(1) = 0.$$

2. (20 points) Obtain the general solution to
$$x\frac{dy}{dx} + 3y = \frac{1+x}{x^3}.$$

3. (20 points) Solve the initial value problem
$$z''(t) - 2z'(t) + 10z(t) = 0, \quad z(0) = 2, \quad z'(0) = -1.$$

4. (20 points) Consider the initial value problem
$$y' = x + 2y^2, \quad y(0) = 1.$$
   (a) (8 points) Use **one step** of Euler's Method to approximate the solution at $x = 1$.
   (b) (12 points) Use **two steps** of Euler's Method to approximate the solution at $x = 1$.

5. (20 points) Pure water flows at a constant rate of 2 L/min into a large tank that initially held 100 L of brine solution containing 4 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3 L/min.
   (a) (5 points) How much solution is in the tank at time t minutes?
   (b) (15 points) Determine the mass of salt in the tank at time t.

1. Separable : $\frac{dy}{dx} = \frac{x^2+1}{x} \cdot \frac{1}{\cos(y)+2e^y}$

$\int (\cos(y) + 2e^y)\, dy = \int \frac{x^2+1}{x}\, dx$

Left $= \sin(y) + 2e^y$ Right $= \int x + \frac{1}{x}\, dx = \frac{1}{2}x^2 + \ln|x|$

$\sin(y) + 2e^y = \frac{1}{2}x^2 + \ln|x| + C$

$x=1, y=0$ $0+2 = \frac{1}{2} + 0 + C$ $C = \frac{3}{2}$ $\boxed{\sin(y)+2e^y = \frac{1}{2}x^2+\ln|x|+\frac{3}{2}}$

2. Linear : $\frac{dy}{dx} + \frac{3}{x}y = \frac{1+x}{x^4}$

$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$ $\frac{d}{dx}[x^3 y] = x^3\left(\frac{1+x}{x^4}\right) = \frac{1+x}{x}$

$x^3 y = \int (\frac{1}{x}+1)\, dx = \ln|x| + x + C$

$y(x) = \frac{1}{x^3}[\ln|x| + x + C] = \boxed{\frac{\ln|x|}{x^3} + \frac{1}{x^2} + \frac{C}{x^3}}$

3. Guess : $z = e^{rt}$ $r^2 - 2r + 10 = 0$ $r = \frac{2 \pm \sqrt{4-4(1)(10)}}{2} = \frac{2\pm\sqrt{-36}}{2}$ $= 1 \pm 3i$

$\alpha = 1, \beta = 6$, $z_1(t) = e^t\cos 3t$, $z_2(t) = e^t \sin 3t$

$z(t) = C_1 e^t \cos 3t + C_2 e^t \sin 3t$, $z' = C_1(e^t\cos 3t + e^t(-3)\sin 3t) + C_2(e^t \sin 3t + 3e^t\cos 3t)$

$2 = z(0) = C_1 + 0$, $C_1 = 2$

$-1 = z'(0) = C_1 + 3C_2 = 2 + 3C_2$, $C_2 = -1$ $\boxed{z(t) = 2e^t\cos 3t - e^t \sin 3t}$

4. (a) $n=1, h=1$ $y_1 = 1 + 1(2) = 3$

n	x	y	$f = x+2y^2$
0	0	1	2
1	1	3	

$y(1) \approx 3$

(b) $n=2$, $h = \frac{1}{2}$

n	x	y	$f = x+2y^2$
0	0	1	2
1	$\frac{1}{2}$	2	$8\frac{1}{2}$
2	1	25/4	

$y_1 = 1 + \frac{1}{2}(2) = 2$

$y_2 = 2 + \frac{1}{2}(8\frac{1}{2})$
$= 2 + 4\frac{1}{4} = 6\frac{1}{4}$

$y(1) \approx 25/4$

5. (a) $V(t) = 100 + (2-3)t = 100 - t$

(b) $\frac{dx}{dt} = 0\frac{kg}{L} \cdot \frac{2L}{min} - \frac{3L}{min}\cdot\frac{x}{100-t}$

$\begin{cases} \frac{dx}{dt} = -\frac{3x}{100-t} \\ x(0) = 4 \end{cases}$ separable

$\int \frac{dx}{x} = \int -3\frac{1}{100-t}\, dt$

$\ln|x| = 3\ln|100-t| + C$

$x(t) = C(100-t)^3$

$x(0) = C(100)^3 = 4$ $C = \frac{4}{10^6}$

$\boxed{x(t) = \frac{4}{10^6}(100-t)^3}$

Fall 2018 Exam 1

DO NOT WRITE ABOVE THIS LINE!!

1. (10 points) Find the explicit form of the solution to

$$2x\frac{dy}{dx} + y = 6x, \quad x > 0, \quad y(4) = 20.$$

$\frac{dy}{dx} + \frac{1}{2x} y = 3$

Factor

$\mu = e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2}\ln x} = x^{\frac{1}{2}}$

Rewriting: $\frac{d}{dx}[x^{\frac{1}{2}} y] = 3x^{\frac{1}{2}}$

Integrate: $x^{\frac{1}{2}} y = 3\int x^{\frac{1}{2}} dx + C$

$x^{\frac{1}{2}} y = 3 \cdot \frac{2}{3} x^{\frac{3}{2}} + C$

$y = 2x + Cx^{-\frac{1}{2}}$

$20 = y(4) = 2\cdot 4 + C \cdot 4^{-\frac{1}{2}}$

$= 8 + \frac{1}{2}C$

$12 = \frac{1}{2}C$

$C = 24$

$y(t) = 2x + 24x^{-\frac{1}{2}}$

2. (10 points) In the year 1955, the US had a population of roughly 170,000,000 people. In 2015, it was estimated to be 340,000,000 people. Using the Malthusian model, $\frac{dP}{dt} = kP$, where k is some constant, find the population P as a function of time t, where 1955 is $t = 0$.

$\frac{dP}{dt} = kP$ separable

$\int \frac{dP}{P} = \int k\,dt$

$\ln P = \cancel{e^{kt}+}\ kt + C$

$p(t) = e^{kt+C} = Ce^{kt}$

$t = 0$ $P(0) = C = 170{,}000{,}000$

2015 $t = 60$ $P(60) = 170.000.000\, e^{k\cdot 60}$

$= 340{,}000{,}000$

$e^{60k} = 2$

$60k = \ln 2$, $k = \frac{\ln 2}{60}$

$P(t) = 170.000.000 \cdot e^{\frac{\ln 2}{60}t}$

DO NOT WRITE ABOVE THIS LINE!!

3. (15 points) (a) Find the homogeneous solution to $y'' + y' - 6y = 0$.

Guess : $y = e^{rt}$

$y_h(t) = C_1 e^{2t} + C_2 e^{-3t}$

a. $r^2 + r - 6 = 0$

$(r-2)(r+3) = 0$

$r_1 = 2, \quad r_2 = -3$

$y_1 = e^{2t} \quad y_2 = e^{-3t}$

(b) Solve the IVP $y'' - 4y' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$.

Guess, $y = e^{rt}$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r_1 = r_2 = 2$ double

$y_1 = e^{2t} \quad y_2 = te^{2t}$

$y(t) = C_1 e^{2t} + C_2 te^{2t}$

$y'(t) = 2C_1 e^{2t} + C_2 e^{2t} + 2C_2 te^{2t}$

$y(0) = C_1 + C_2(0) = 1 \quad C_1 = 1$

$y'(0) = 2C_1 + C_2 + 0 = 2 + C_2 = 0$

$C_2 = -2$

$y(t) = e^{2t} - 2te^{2t}$

4. (15 points) (a) Use Euler's method with two steps to estimate $y(1)$ if $y(x)$ is the solution to

$$\frac{dy}{dx} = x + y, \quad y(0) = 1.$$

Write your answer in the form $\frac{a}{b}$ where a and b are integers.

$n=2$, $h = \frac{1-0}{2} = \frac{1}{2}$ $\quad x_0 = 0$, $y_0 = 1$

n	x	y	f=x+y
0	0	1	1
1	$\frac{1}{2}$	$\frac{3}{2}$	2
2	1	$\frac{5}{2}$	$\frac{7}{2}$

$y(1) \approx \frac{5}{2}$

$y_1 = 1 + \frac{1}{2}(1) = \frac{3}{2}$

$y_2 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$

(b) What is the exact value of $y(1)$ if $y(x)$ is the solution to the IVP above?

$\frac{dy}{dx} - 1y = x$

$\mu = e^{\int -1 dx} = e^{-x}$

$\frac{d}{dx}[e^{-x}y] = xe^{-x}$

$e^{-x}y = \int xe^{-x}dx + C$

$u = x \qquad du = dx$

$dv = e^{-x}dx \qquad v = -e^{-x}$

$= -xe^{-x} - \int -e^{-x}dx + C$

$= -xe^{-x} - e^{-x} + C$

$y(x) = -x - 1 + Ce^{x}$

$y(0) = 0 - 1 + C = 1$

$C = 2$

$y(x) = -x - 1 + 2e^{x}$

$y(1) = -2 + 2e$

DO NOT WRITE ABOVE THIS LINE!!

5. (20 points) Consider the differential equation $y'' + 9y = f(t)$.

(a) Solve the equation if $f(t) = 0$; that is, if the differential equation is homogeneous.

Guess: $\cancel{r^2+9=0}$ $y = e^{rt}$

$r^2 + 9 = 0 \qquad r = 0 \pm 3i$

$y_1 = \cos 3t \qquad y_2 = \sin 3t$

$y(t) = C_1 \cos 3t + \cancel{y_2}\, C_2 \sin 3t$

(b) Find the form of a particular solution if $f(t) = t^2 \cos(3t)$. Do not evaluate the coefficients.

~~Ch3~~. 4.4 or later

(c) Find the form of a particular solution if $f(t) = e^{4t}$. In this part you SHOULD evaluate the coefficients.

4.4 or later