

10.2 Method of Separation of Variables

1. The heat flow problem or Heat equation

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \beta \frac{\partial^2 u(x, t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0, t) = u(L, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < L \end{cases}$$

2. Determine all solutions to the boundary value problem
 $y'' + y = 0, \quad 0 < x < 2\pi; \quad y(0) = 0, \quad y(2\pi) = 1$

3. Find the values of λ (eigenvalues) for which the boundary value problem has a nontrivial solution. Then determine the corresponding nontrivial solutions (eigenfunctions).
- (a) $y'' + \lambda y = 0, \quad 0 < x < \pi; \quad y(0) = 0, \quad y'(\pi) = 0$
 - (b) $y'' + \lambda y = 0, \quad 0 < x < \pi; \quad y'(0) = 0, \quad y(\pi) = 0$
 - (c) $y'' + \lambda y = 0, \quad 0 < x < 2\pi; \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$

4. Solve the heat flow problem

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = \beta \frac{\partial^2 u(x, t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0, t) = u(L, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < L \end{cases}$$

with $\beta = 3$, $L = \pi$, and the given function $f(x) = \sin 3x + 5 \sin 7x - 2 \sin 13x$.

5. Solve the vibrating string problem

$$\begin{cases} \frac{\partial^2 u(x, t)}{\partial t^2} = \alpha^2 \frac{\partial^2 u(x, t)}{\partial x^2}, & 0 < x < L, t > 0 \\ u(0, t) = u(L, t) = 0, & t \geq 0 \\ u(x, 0) = f(x), & 0 \leq x \leq L \\ \frac{\partial u}{\partial t}(x, 0) = g(x), & 0 \leq x \leq L \end{cases}$$

with $\alpha = 3$, $L = \pi$, and the given initial functions

$$f(x) = \sin x - 2 \sin 2x + \sin 3x, \quad g(x) = 6 \sin 3x - 7 \sin 5x.$$