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A REMARK ON MODULI OF COMPLEX HYPERSURFACES

By Stephen S.-T. Yau*

1. Statement of results. The purpose of this short note is to point out an application of Mather-Yau Theorem [3] in complex algebraic geometry. One of the fundamental problems in complex algebraic geometry is to find complete continuous invariants explicitly for any given family of varieties so that two varieties in this family are isomorphic if and only if they have the same invariants. Our main theorem allows us to reduce the problem in some cases to a family of lower dimensional varieties.

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. We say that f is a quasi-homogeneous function if f is in the ideal

$$\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_{\mathbf{C}^n,0}$$

generated by the partial derivatives of f in the local ring $\mathcal{O}_{\mathbf{C}^n,0}$. Saito in [5] proved that, after a biholomorphic change of variables, f is a weighted homogeneous polynomial.

Definition. Let $(X_1, 0)$ and $(X_2, 0)$ be two isolated hypersurface singularities in $(\mathbf{C}^n, 0)$. We say that $(X_1, 0)$ and $(X_2, 0)$ have the same analytic type (respectively topological type) if there exists a germ of biholomorphism (resp. homeomorphism) from $(\mathbf{C}^n, X_1, 0)$ to $(\mathbf{C}^n, X_2, 0)$.

In view of the results of [8], the following question was pointed out to me by Lê Dũng Tráng.

Question. Let $f(z_1, \ldots, z_n) = 0$ and $h(w_1, \ldots, w_m) = 0$ be the defining equations for isolated hypersurface singularities $(X_f, 0) \subseteq (\mathbf{C}^n,$

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0) and $(X_h, 0) \subseteq (\mathbf{C}^m, 0)$. Does the topological type of the hypersurface X_{f+h} defined by $f(z_1, \ldots, z_n) + h(w_1, \ldots, w_m) = 0$ (addition of Thom-Sebastiani) in $(\mathbf{C}^{n+m}, 0)$ depend only on the topological type of $(X_f, 0)$ and $(X_h, 0)$?

In his 1977 paper [8], Teissier introduced the concept for two isolated hypersurface singularities being (c)-cosécantes. He showed that the (c)-cosécantes class of the hypersurface defined by $f(z_1, \ldots, z_n) + h(w_1, \ldots, w_m) = 0$ depends only on the (c)-cosécantes class of $(X_f, 0)$ and $(X_h, 0)$. He remarked that the analytic type of the hypersurface X_{f+h} defined by $f(z_1, \ldots, z_n) + h(w_1, \ldots, w_m) = 0$ depends not only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$, but also in general on the choice of the equation for f and h. However the following theorem says that in case h is quasi-homogeneous, then the analytic type of X_{f+h} indeed depends only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$. In fact, a "subtraction" theorem holds!

THEOREM. *Let $f(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$ be holomorphic functions with isolated singularity at origin in $\mathbf{C}^n$, and $h(w_1, \ldots, w_m)$ be a quasi-homogeneous holomorphic function with an isolated singularity at origin. Then $(X_f, 0)$ is biholomorphically equivalent to $(X_g, 0)$ if and only if $(X_{f+h}, 0)$ is biholomorphically equivalent to $(X_{g+h}, 0)$.*

As a typical application of the above theorem, we have the following examples.

Example 1. Let $\mathcal{M}_{d,n}$ be the moduli space of nonsingular hypersurfaces of degree d in $\mathbf{P}^n$. Then there is a canonical injection from the moduli space $\mathcal{M}_{d,2}$ of nonsingular curves of degree d in $\mathbf{P}^2$ into $\mathcal{M}_{d,n}$. In particular, $\mathcal{M}_{4,2}$, which is a Zariski dense open subset of the moduli space $\mathcal{M}_3$ of complete curves of genus 3 is mapped injectively into $\mathcal{M}_{4,n}$ for $n \geq 3$.

Example 2. Let $V_t = \{(z_0, z_1, z_2, \ldots, z_n) \in \mathbf{C}^{n+1} : z_0^4 + tz_0^2z_1^2 + z_1^4 + g(z_2, \ldots, z_n) = 0$ where $t^2 \neq 4$ and $g(z_2, \ldots, z_n)$ is a quasi-homogeneous holomorphic function with isolated singular point at $(z_2, \ldots, z_n) = (0, \ldots, 0)\}$. Then the complete continuous invariant of this one parameter family is given by $c(t) = (t^2 + 12)^3/(t^2 - 4)^2$ i.e. V_{t_1} is not biholomorphically equivalent to V_{t_2} if and only if $c(t_1) \neq c(t_2)$.

Example 3. Let $V_t = \{(z_0, z_1, z_2, \ldots, z_n) \in \mathbf{C}^{n+1} : z_0^3 + z_1^3 + z_2^3 + tz_0z_1z_2 + g(z_3, \ldots, z_n) = 0$ where $t^3 + 27 \neq 0$ and $g(z_3, \ldots, z_n)$ are quasi-homogeneous holomorphic functions with isolated singular points at $(z_3, z_4, \ldots, z_n) = (0, 0, \ldots, 0)\}$. Then the complete invariant of this one parameter family is given by $c(t) = [t(t^3 - 216)/(t^3 + 27)]^3$.

Example 4. Let $V_t = \{(z_0, z_1, z_2, \ldots, z_n) \in \mathbf{C}^{n+1} : z_0^6 + z_1^3 + tz_0^4 z_1 + g(z_2, \ldots, z_n) = 0$ where $4t^3 + 27 \neq 0$ and $g(z_2, \ldots, z_n)$ is a quasi-homogeneous holomorphic function with isolated singular points at $(z_2, \ldots, z_n) = (0, \ldots, 0)\}$. Then the complete invariant of this one parameter family is given by $c(t) = t^3$.

2. Proof of the theorem and examples.

Proof of the theorem. We first prove that if $(X_f, 0)$ is biholomorphically equivalent to $(X_g, 0)$, then $(X_{f+h}, 0)$ is biholomorphically equivalent to $(X_{g+h}, 0)$. By the theorem of Mather-Yau [3], it is equivalent to show that $A(X_{f+h})$ is isomorphic to $A(X_{g+h})$ as a $\mathbf{C}$-algebra. Recall that

$$\begin{aligned}
A(X_{f+h}) := {} & \mathbf{C}\{z_1, \ldots, z_{n+k}\}\Big/\Big(f(z_1, \ldots, z_n) \\
& + h(z_{n+1}, \ldots, z_{n+k}), \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial h}{\partial z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\Big) \\
= {} & \mathbf{C}\{z_1, \ldots, z_{n+k}\}\Big/\Big(f(z_1, \ldots, z_n), \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \\
& \frac{\partial h}{\partial z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\Big) \\
\cong {} & \mathbf{C}\{z_1, \ldots, z_n\}\Big/\Big(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\Big) \\
& \otimes \mathbf{C}\{z_{n+1}, \ldots, z_{n+k}\}\Big/\Big(\frac{\partial h}{\partial z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\Big).
\end{aligned}$$

The second equality above comes from the fact that h is quasi-homogeneous while the last isomorphism follows from the Korollar 1 and Korollar 2 of p. 181 in [1]. Similarly we have

$$A(X_{g+h}) \simeq \mathbf{C}\{z_1, \ldots, z_n\}/\left(g, \frac{\partial g}{\partial z_n}, \ldots, \frac{\partial g}{\partial z_n}\right)$$

$$\otimes\ \mathbf{C}\{z_{n+1}, \ldots, z_{n+k}\}/\left(\frac{\partial h}{z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\right).$$

However, since $(X_f, 0)$ is biholomorphically equivalent to $(X_g, 0)$, we have

$$\mathbf{C}\{z_1, \ldots, z_n\}/\left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

$$\cong \mathbf{C}\{z_1, \ldots, z_n\}/\left(g, \frac{\partial g}{\partial z_1}, \ldots, \frac{\partial g}{\partial z_n}\right)$$

by the theorem of Mather-Yau [3] again. Therefore we conclude that $A(X_{f+h}) \cong A(X_{g+h})$.

Conversely, suppose that $(X_{f+h}, 0)$ is biholomorphically equivalent to $(X_{g+h}, 0)$. By the theorem of Mather-Yau [3], the moduli algebra $A(X_{f+g})$ is isomorphic to the moduli algebra of $A(X_{g+h})$. Therefore we have

$$\mathbf{C}\{z_1, \ldots, z_n\}/\left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

$$\otimes\ \mathbf{C}\{z_{n+1}, \ldots, z_{n+k}\}/\left(\frac{\partial h}{\partial z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\right)$$

$$\cong \mathbf{C}\{z_1, \ldots, z_n\}/\left(g, \frac{\partial g}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$$

$$\otimes\ \mathbf{C}\{z_{n+1}, \ldots, z_{n+k}\}/\left(\frac{\partial h}{\partial z_{n+1}}, \ldots, \frac{\partial h}{\partial z_{n+k}}\right).$$

It follows from the cancellation theorem for Artinian local algebras [2] that $A(X_f) = \mathbf{C}\{z_1, \ldots, z_n\}/(f, \partial f/\partial z_1, \ldots, \partial f/\partial z_n) \cong \mathbf{C}\{z_1, \ldots, z_n\}/(g, \partial g/\partial z_1, \ldots, \partial g/\partial z_n) = A(X_g)$. Apply the theorem of Mather-Yau again to conclude that $(X_f, 0)$ is biholomorphically equivalent to $(X_g, 0)$. Q.E.D.

Proof of the Example 1. Let $f_t(z_0, z_1, z_2) = t_1 z_0^d + t_2 z_0^{d-1} z_1 + \cdots + t_l z_2^d$ be a nonsingular hypersurface of degree d in $\mathbf{P}^2$ where $l =$ number of monomials of degree d in 3 variables. Let $F_t(z_1, z_1, \ldots, z_n) = f_t(z_1, z_1, z_2) + z_3^d + z_4^d + \cdots + z_n^d$. Then $F_t(z_1, z_1, \ldots, z_n)$ is a nonsingular hypersurface of degree d in $\mathbf{P}^n$. By the main theorem, $V_t = \{(z_0, z_1, z_2) \in \mathbf{P}^2 : f_t(z_0, z_1, z_2) = 0\}$ is biholomorphically equivalent to $V_{t'} = \{(z_0 : z_1 : z_2) \in \mathbf{P}^2 : f_{t'}(z_0, z_1, z_2) = 0\}$ if and only if $W_t = \{(z_0 : z_1 : \cdots : z_n) \in \mathbf{P}^n : F_t(z_0, z_1, \ldots, z_n) = 0\}$ is biholomorphically equivalent to $W_{t'} = \{(z_0 : z_1 : \cdots : z_n) \in \mathbf{P}^n : F_{t'}(z_0, z_1, \ldots, z_n) = 0\}$. Example 1 follows immediately.

Proof of Examples 2, 3 *and* 4. It is well known that the complete continuous invariant for the family $W_t = \{(z_0, z_1, z_2) \in \mathbf{C}^3 : z_0^3 + z_1^3 + z_2^3 + tz_0z_1z_2 = 0\}$ is given by $C(t) = t^3(t^3 - 216)^3/(t^3 + 27)^3$ while the complete continuous invariant for the family $\{(z_0, z_1) \in \mathbf{C}^2 : z_0^4 + tz_0^2z_1^2 + z_1^4 = 0\}$ is given by $c(t) = (t^2 + 12)^3/(t^2 - 4)^2$. In [7], the complete continuous invariant for the family $\{(z_0, z_1) \in \mathbf{C}^2 : z_0^6 + z_1^3 + tz_0^4z_1\}$ was computed via the Lie algebra associated to this family of isolated singularities. It is given by $c(t) = t^3$. Now the Examples 2, 3 and 4 follow immediately from the theorem. Q.E.D.

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