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Classification of Isolated Hypersurface Singularities by Their Moduli Algebras

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We will prove that two germs of complex analytic hypersurfaces with isolated singularities are biholomorphically equivalent if and only if they have the same dimension and their moduli algebras are isomorphic. This result was announced in [1]. Examples given in [1] show that the analogous statements over the real numbers and over all fields of characteristic $p>0$ are false.

§ 1. Biholomorphic Equivalence of Isolated Hypersurface Singularities

Let $\mathcal{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $(\mathbb{C}^{n+1},0)\to\mathbb{C}$. If $(V,0)$ is a germ at the origin of a hypersurface in $\mathbb{C}^{n+1}$, let $I(V)$ be the ideal of functions in $\mathcal{O}_{n+1}$ vanishing on V, and let f be a generator of $I(V)$. It is well known that $V-0$ is non-singular if and only if the $\mathbb{C}$-vector space

$$A(V)=\mathcal{O}_{n+1}\Big/\left(f,\frac{\partial f}{\partial z_0},\ldots,\frac{\partial f}{\partial z_n}\right)\mathcal{O}_{n+1}$$

is finite dimensional. The germ at 0 of $A(V)$ is the base space for the miniversal deformation of $(V,0)$, so it seems natural to call $A(V)$, provided with the obvious $\mathbb{C}$-algebra structure, the *moduli algebra* of V. Our theorem will state that V is determined by $A(V)$ and also by the following $\mathbb{C}$-algebra:

$$B(V)=\mathcal{O}_{n+1}\Big/\left(f,z_i\frac{\partial f}{\partial z_j}\right)\mathcal{O}_{n+1}\qquad 1\leqq i,\ j\leqq n.$$

Theorem. *Suppose $(V,0)$ and $(W,0)$ are germs of hypersurfaces in $\mathbb{C}^{n+1}$, and $V-0$ is non-singular. Then the following conditions are equivalent.*

(i) *$(V,0)$ is biholomorphically equivalent to $(W,0)$.*
(ii) *$A(V)$ is isomorphic to $A(W)$ as a $\mathbb{C}$-algebra.*
(iii) *$B(V)$ is isomorphic to $B(W)$ as a $\mathbb{C}$-algebra.*

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Remark. This theorem was suggested to us by an earlier result of Max Benson. Using a different method, he proved the above theorem in the case V is a homogeneous hypersurface.

§ 2. Proof of (i) ⇒ (ii) and (i) ⇒ (iii)

Let f and g be generators of $I(V)$ and $I(W)$, respectively. Let $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ be the germ at the origin of a biholomorphic mapping, such that $h(V)=W$. Then there exists $u \in \mathcal{O}_{n+1}$ such that $f = u \cdot (g \circ h)$ and $u(0) \neq 0$. Write $h=(h_0, \ldots, h_n)$, where $h_i: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then

$$\frac{\partial f}{\partial z_i} = \frac{\partial u}{\partial z_i} (g \circ h) + u \sum_{j=0}^{n} \left(\frac{\partial g}{\partial z_j} \circ h\right) \frac{\partial h}{\partial z_i}.$$

Hence, $\frac{\partial f}{\partial z_i}$ is in the ideal generated by $g \circ h$, $\frac{\partial g}{\partial z_0} \circ h, \ldots, \frac{\partial g}{\partial z_n} \circ h$. A similar argument shows that $\frac{\partial g}{\partial z_i}$ is in the ideal generated by

$$f \circ h^{-1}, \frac{\partial f}{\partial z_0} \circ h^{-1}, \ldots, \frac{\partial f}{\partial z_n} \circ h^{-1}.$$

From this, it follows immediately that

$$h^* \left(g, \frac{\partial g}{\partial z_0}, \ldots, \frac{\partial g}{\partial z_n}\right) \mathcal{O}_{n+1} = \left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{n+1}$$

and

$$h^* \left(g, z_i \frac{\partial g}{\partial z_j}\right) \mathcal{O}_{n+1} = \left(f, z_i \frac{\partial f}{\partial z_j}\right) \mathcal{O}_{n+1},$$

where $h^*: \mathcal{O}_{n+1} \to \mathcal{O}_{n+1}$ is the $\mathbb{C}$-algebra isomorphism defined by $h^* u = u \circ h$. It follows that h^* induces $\mathbb{C}$-algebra isomorphisms $A(V) \approx A(W)$ and $B(V) \approx B(W)$.

§ 3. The Group $\mathcal{K}$

In [2, § 2], the first author defined a group $\mathcal{K}$ of germs of C^∞ mappings associated to a pair (n, p) of positive integers. In this section we will define an analogous group $\mathcal{K}$ of holomorphic mappings associated to $(n+1, 1)$. Then $\mathcal{K}$ acts on $\mathcal{O}_{n+1}$. If $(V, 0)$ and $(W, 0)$ are germs at the origin of hypersurfaces, and $I(V) = f\mathcal{O}_{n+1}$, $I(W) = g\mathcal{O}_{n+1}$, then $(V, 0)$ and $(W, 0)$ are biholomorphically equivalent if and only if f and g lie in the same $\mathcal{K}$-orbit. These assertions may be proved in exactly the same way as the analogous assertions in [2] were proved. However, we will review parts of the proofs here to avoid any possible confusion due to changes in notation, etc.

Definition of $\mathcal{K}$. By $\mathcal{K}$, we will mean the set of pairs (h, H) such that $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ and $H: (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}^{n+2}, 0)$ are germs of biholomorphic mappings such that the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{C}^{n+1}, 0) & \xrightarrow{i} & (\mathbb{C}^{n+2}, 0) & \xrightarrow{\pi} & (\mathbb{C}^{n+1}, 0) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h} \\ (\mathbb{C}^{n+1}, 0) & \xrightarrow{i} & (\mathbb{C}^{n+2}, 0) & \xrightarrow{\pi} & (\mathbb{C}^{n+1}, 0), \end{array}$$

where $i(z_0, \ldots, z_n) = (z_0, \ldots, z_n, 0)$ and $\pi(z_0, \ldots, z_n, w) = (z_0, \ldots, z_n)$. We define the group structure by $(h, H)(h_1, H_1) = (h \circ h_1, H \circ H_1)$.

The Action of $\mathcal{K}$ on $\mathcal{O}_{n+1}$. Given $(h, H) \in \mathcal{K}$ and $f \in \mathcal{O}_{n+1}$, we define $g = (h, H) \cdot f$ to be the unique element of $\mathcal{O}_{n+1}$ such that

$$\operatorname{graph} g = H(\operatorname{graph} f).$$

Biholomorphic Equivalence of Hypersurfaces. Let $(V, 0)$ and $(W, 0)$ be germs of hypersurfaces in $\mathbb{C}^{n+1}$ and let f and g be generators of $I(V)$ and $I(W)$, respectively.

Proposition 3.1. *f and g are in the same $\mathcal{K}$-orbit if and only if $(V, 0)$ and $(W, 0)$ are biholomorphically equivalent.*

Proof. "Only if." Let $(h, H) \in \mathcal{K}$ be such that $H(\operatorname{graph} f) = \operatorname{graph} g$. Then $h^{-1}(W) = h^{-1}(i^{-1} \operatorname{graph} g) = i^{-1}(H^{-1} \operatorname{graph} g) = i^{-1}(\operatorname{graph} f) = V$. (These equations are equalities between germs of sets.) Hence, h provides a biholomorphic equivalence between V and W.

"If." Let $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ be a germ of a biholomorphic mapping such that $h(V) = W$. Then there is a unit $u \in \mathcal{O}_{n+1}$ such that $f = u \cdot (g \circ h)$. Define $H: (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}^{n+2}, 0)$ by

$$H(z, w) = (h(z), u^{-1}(z)\, w), \qquad \text{for } z \in \mathbb{C}^{n+1},\ w \in \mathbb{C}.$$

Then $(h, H) \in \mathcal{K}$, and $H(z, f(z)) = (h(z), u^{-1}(z) \cdot f(z)) = (h(z), g(h(z)))$, so $H(\operatorname{graph} f) = \operatorname{graph} g$. $\square$

Let J^k denote the $\mathbb{C}$-vector space of k-jets at the origin of elements of $\mathcal{O}_{n+1}$. Let $\mathcal{K}^k$ denote the Lie group of k-jets at the origin of members of $\mathcal{K}$. Since $\mathcal{K}$ acts on $\mathcal{O}_{n+1}$, we have that $\mathcal{K}^k$ acts on J^k. For $f \in \mathcal{O}_{n+1}$, let $f^{(k)} \in J^k$ denote the k-jet of f at the origin. We say f is k-determined with respect to $\mathcal{K}$ if $g \in \mathcal{O}_{n+1}$ and $g^{(k)} \in \mathcal{K}^k f^{(k)}$ imply $g \in \mathcal{K} f$. We say f is *finitely determined* with respect to $\mathcal{K}$ if it is k-determined with respect to $\mathcal{K}$ for some positive integer k.

Proposition 3.2. *Let $(V, 0)$ be a germ of a hypersurface in $\mathbb{C}^{n+1}$ and let f be a generator if $I(V)$. Then the following are equivalent:*

a) *$V - 0$ is non-singular.*
b) *$A(V)$ is finite dimensional as a $\mathbb{C}$-vector space.*
c) *$B(V)$ is finite dimensional as a $\mathbb{C}$-vector space.*
d) *f is finitely determined with respect to $\mathcal{K}$.*

The equivalence of a), b), and c) is well known. The equivalence of d) to b) is a special case of the complex analogue of [2, Theorem 3.5]. The complex analogue of [2, Theorem 3.5] can be proved in exactly the same way as [2, Theorem 3.5]. To formulate the special case we use of the complex analogue of [2, Theorem 3.5], it is necessary to replace the $\mathbb{R}$-algebra $C(N)_S$ by the $\mathbb{C}$-algebra $\mathcal{O}_{n+1}$, the $C(N)_S$ module B by the $\mathcal{O}_{n+1}$-module consisting of germs at the origin of complex analytic vector fields on $\mathcal{O}_{n+1}$, and the $C(N)_S$-module $\theta(f)$ by $\mathcal{O}_{n+1}$. In our situation, we have an isomorphism of $\mathbb{C}$-vector spaces:

$$A(V) \approx \theta(f)/(tf(B)+f^*(\mathfrak{m}_y))\,\theta(f).$$

§4. Proof of (iii)$\Rightarrow$(i): Reduction to a Special Case

In this section, we will show that it is enough to prove (i) under the hypothesis that Eq. (4) below holds. We may suppose that $f^{(1)}=0$, for otherwise both V and W are non-singular at the origin, and the implication (iii)$\Rightarrow$(i) is obvious.

Let $\varphi: B(V) \to B(W)$ be a $\mathbb{C}$-algebra isomorphism. We denote the coordinate functions on $\mathbb{C}^{n+1}$ by $z_0, \ldots, z_n$. These are elements of the maximal ideal of $\mathcal{O}_{n+1}$. We let $[z_0], \ldots, [z_n]$ denote the images of $z_0, \ldots, z_n$ in $B(V)$. From the hypothesis that $f^{(1)}=0$, it follows that $[z_0], \ldots, [z_n]$ are linearly independent modulo the square of the maximal ideal in $B(V)$. Hence $\varphi([z_0]), \ldots, \varphi([z_n])$ are linearly independent modulo the square of the maximal ideal in $B(W)$. Let $h_0, \ldots, h_n$ be elements of $\mathcal{O}_{n+1}$ which project onto $\varphi([z_0]), \ldots, \varphi([z_n])$. Then $h_0, \ldots, h_n$ are in the maximal ideal of $\mathcal{O}_{n+1}$ and are linearly independent modulo the square of the maximal ideal in $\mathcal{O}_{n+1}$. It follows that $h=(h_0, \ldots, h_n): (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ is a germ of a biholomorphic mapping. We have that the diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{h^*} & \mathcal{O}_{n+1} \\ \downarrow & & \downarrow \\ B(V) & \xrightarrow{\varphi} & B(W) \end{array}$$

commutes. In other words,

$$\begin{aligned} \left(g, z_i \frac{\partial g}{\partial z_j}\right) \mathcal{O}_{n+1} &= h^* \left(f, z_i \frac{\partial f}{\partial z_j}\right) \mathcal{O}_{n+1} \\ &= \left(f \circ h, h_i \frac{\partial f}{\partial z_j} \circ h\right) \mathcal{O}_{n+1} = \left(f \circ h, z_i \frac{\partial f}{\partial z_j} \circ h\right) \mathcal{O}_{n+1} \\ &= \left(f \circ h, z_i \frac{\partial (f \circ h)}{\partial z_j}\right) \mathcal{O}_{n+1}. \end{aligned}$$

The third equality here comes from the fact that there is an invertible matrix (a_{ij}) of elements of $\mathcal{O}_{n+1}$ such that $z_i = \sum a_{ij} h_j$. The fourth equality comes from the chain rule $\frac{\partial (f \circ h)}{\partial z_j} = \sum_k \frac{\partial f}{\partial z_k} \circ h \frac{\partial h_k}{\partial z_j}$ and the fact that the matrix $\frac{\partial h_k}{\partial z_j}$ is invertible.

By replacing f with $f \circ h$, we may therefore assume

$$\left(f, z_i \frac{\partial f}{\partial z_j}\right) \mathcal{O}_{n+1} = \left(g, z_i \frac{\partial g}{\partial z_j}\right) \mathcal{O}_{n+1}. \quad \square \tag{4}$$

§ 5. Definition and Properties of L_0

The hypothesis that $V - 0$ is non-singular in our theorem and the implication a)⇒d) in Proposition 3.2 imply that f is finitely determined with respect to $\mathcal{K}$. Hence it is enough to prove that $g^{(k)} \in \mathcal{K}^{(k)} f^{(k)}$, for every positive integer k. In what follows, we let k be a fixed positive integer.

Obviously, there is nothing to prove when $f^{(k)} = g^{(k)}$, so we suppose $f^{(k)} \neq g^{(k)}$. We let L be the complex line in J^k containing $f^{(k)}$ and $g^{(k)}$. We let L_0 be the set of $h^{(k)}$ in L such that

$$\left(h, z_i \frac{\partial h}{\partial z_j}\right) J^k = \left(f, z_i \frac{\partial f}{\partial z_j}\right) J^k. \tag{5}$$

Here, we think of J^k as a module over $\mathcal{O}_{n+1}$, so the two sides of (5) are submodules of J^k. The left side depends only on $h^{(k)}$.

Let M denote the set of multi-indices α such that $|\alpha| \leqq k$ i.e., the set of $(n+1)$-tuples $(\alpha_0, \ldots, \alpha_n)$ of non-negative integers, such that $\alpha_0 + \ldots + \alpha_n \leqq k$. Let M_1 denote the subset of M consisting of α for which $|\alpha| > 0$. Let N be the number of elements in $M \amalg (M_1 \times \{0, \ldots, n\})$. (The symbol $\amalg$ means "disjoint union.") Let $\varphi: \{1, \ldots, N\} \to M \amalg (M_1 \times \{0, \ldots, n\})$ be a bijection. Consider $w \in \mathbb{C}$, and set $h = (1-w) f + w g$. Then $h^{(k)} \in L$. Let $v_i(w) = (z^{\varphi(i)} h)^{(k)}$, if $\varphi(i) \in M$, and

$$v_i(w) = \left(z^{\varphi(i)(1)} \frac{\partial h}{\partial z_{\varphi(i)(2)}}\right)^{(k)},$$

if $\varphi(i) \in M_1 \times \{0, \ldots, n\}$, where we denote the components of $\varphi(i)$ by $\varphi(i)(1)$ and $\varphi(i)(2)$.

We have that the left side of (5) is spanned over $\mathbb{C}$ by $v_1(w), \ldots, v_N(w)$. Moreover, $v_i(w) = (1-w) v_i(0) + w v_i(1)$, and the left side of (5) equals the right side when $w = 0$ or 1, by (4).

It follows that the left side of (5) is contained in the right side, for all w. Let d denote the dimension of the right side of (5), considered as a $\mathbb{C}$-vector space. By choosing a basis of the right side of (5), we may represent each $v_i(w)$ as a row vector of length d. Then the $v_i(w)$'s form the rows of an $N \times d$ matrix $V(w)$ of complex numbers.

The Eq. (5) holds if and only if at least one of the $d \times d$ minor determinants of this matrix is nonzero. Since $v_i(w) = (1-w) v_i(0) + w v_i(1)$, at least one of these minor determinants does not vanish identically. Hence there are at most d points w in $\mathbb{C}$ for which (5) does not hold.

We have just shown that L_0 is L with at most finitely many points deleted. Since L is a complex line, it follows that L_0 is a connected manifold.

§6. End of the Proof that (iii)⇒(i)

By what we have done above, it is enough to show that (4) implies $g^{(k)} \in \mathscr{K}^k f^{(k)}$. We will show this by an application of [3, Lemma 3.1]. For this purpose, we take the action α of [3, Lemma 3.1] to be the action of $\mathscr{K}^k$ on J^k which we have been discussing. We take the submanifold V of [3, Lemma 3.1] to be L_0. By the previous section L_0 is a connected C^∞ submanifold of J^k.

To verify condition a) of [3, Lemma 3.1], we observe that $T(\mathscr{K}^k[h])_{[h]} = \left(h, z_i \frac{\partial h}{\partial z_j}\right) J^k$, for any $h \in \mathcal{O}_{n+1}$, where $[h]=h^{(k)}$. This is a special case of the complex analogue of [2, Proposition 7.4a]. If $[h] \in L_0$, then (5) holds, and we obtain

$$T(\mathscr{K}^k[h]) = \left(f, z_i \frac{\partial f}{\partial z_j}\right) J^k. \tag{6}$$

Obviously, $T(L_0)_{[h]}$ is the one dimensional complex subspace of J^k spanned by $g-f$. By (4), $g-f \in \left(f, z_i \frac{\partial f}{\partial z_j}\right) J^k$. Hence $T(L_0)_{[h]} \subset T(\mathscr{K}^k[h])_{[h]}$, i.e., condition a) of [3, Lemma 3.1] holds.

Condition b) of [3, Lemma 3.1] follows immediately from (6). Hence, L_0 is contained in a single orbit of the action of $\mathscr{K}^k$ on J^k, by [3, Lemma 3.1]. In particular, $g^{(k)} \in \mathscr{K}^k f^{(k)}$.

§7. Proof that (ii)⇒(iii). Reduction to a Special Case

Let $\varphi: A(V) \to A(W)$ be a $\mathbb{C}$-algebra isomorphism. In this section, we let $z_0, \ldots, z_n$ be a holomorphic local system of coordinates on $\mathbb{C}^{n+1}$, centered at the origin. We let $[z_0], \ldots, [z_n]$ be the images of $z_0, \ldots, z_n$ under the projection $\mathcal{O}_{n+1} \to A(V)$. Let K denote the kernel of this projection and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{n+1}$. Let

$$k = \dim_{\mathbb{C}} \frac{K \cap \mathfrak{m} + \mathfrak{m}^2}{\mathfrak{m}^2}.$$

We may choose the system of coordinates $z_0, \ldots, z_n$ in the following way. First, we choose $z_0, \ldots, z_{k-1} \in K \cap \mathfrak{m}$ which are linearly independent $\bmod \mathfrak{m}^2$. Then we choose $z_k, \ldots, z_n \in \mathfrak{m}$ such that $z_0, \ldots, z_n$ are linearly independent $\bmod \mathfrak{m}^2$. By the inverse function theorem, $z_0, \ldots, z_n$ form a holomorphic local system of coordinates on $\mathbb{C}^{n+1}$, centered at the origin.

Let $K' = \ker(\mathcal{O}_{n+1} \to A(W))$. Since $A(V)$ and $A(W)$ are isomorphic as $\mathbb{C}$-algebras, we have that

$$k = \dim_{\mathbb{C}} \frac{K' \cap \mathfrak{m} + \mathfrak{m}^2}{\mathfrak{m}^2}.$$

Hence, we may choose elements $w_0, \ldots, w_{k-1} \in K' \cap \mathfrak{m}$ which are linearly independent modulo $\mathfrak{m}^2$. For $i \geq k$, we let w_i be an element of $\mathfrak{m}$ which projects onto $\varphi([z_i])$ under the projection $\mathcal{O}_{n+1} \to A(W)$. Since $[z_k], \ldots, [z_n]$ are linearly independent modulo the square of the maximal ideal in $A(V)$, we have that $\varphi([z_k]), \ldots, \varphi([z_n])$ are linearly independent modulo the square of the maximal ideal in $A(W)$. Hence $w_k, \ldots, w_n$ are linearly independent modulo $K' \cap \mathfrak{m} + \mathfrak{m}^2$, and $w_0, \ldots, w_n$ are linearly independent modulo $\mathfrak{m}^2$. By the inverse function theorem, $w_0, \ldots, w_n$ therefore form a local system of coordinates for $\mathbb{C}^{n+1}$, centered at the origin.

We define a germ $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ of a biholomorphic mapping by $z_i \circ h = w_i$. Letting $\bar{w}_i$ denote the image of w_i under the projection $\mathcal{O}_{n+1} \to A(W)$, we have $\varphi([z_i]) = \bar{w}_i$, $i = 0, \ldots, n$. (For $0 \leqq i \leqq k-1$, $[z_i] = 0$ and $\bar{w}_i = 0$.) Hence the diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{h^*} & \mathcal{O}_{n+1} \\ \downarrow & & \downarrow \\ A(V) & \xrightarrow{\varphi} & A(W) \end{array}$$

commutes. In other words,

$$\left(g, \frac{\partial g}{\partial z_i}\right) \mathcal{O}_{n+1} = h^* \left(f, \frac{\partial f}{\partial z_i}\right) \mathcal{O}_{n+1}$$

$$= \left(f \circ h, \frac{\partial f}{\partial z_i} \circ h\right) \mathcal{O}_{n+1} = \left(f \circ h, \frac{\partial (f \circ h)}{\partial z_i}\right) \mathcal{O}_{n+1}.$$

The last equality comes from the chain rule $\dfrac{\partial (f \circ h)}{\partial z_i} = \sum_j \left(\dfrac{\partial f}{\partial z_j} \circ h\right) \dfrac{\partial h_j}{\partial z_i}$ and the fact that the matrix $\left(\dfrac{\partial h_j}{\partial z_i}\right)$ is invertible.

By replacing f with $f \circ h$, we may therefore assume

$$\left(f, \frac{\partial f}{\partial z_i}\right) \mathcal{O}_{n+1} = \left(g, \frac{\partial g}{\partial z_i}\right) \mathcal{O}_{n+1}. \quad \square \tag{7}$$

§8. End of the Proof that (ii)⇒(iii)

By the previous section, we may assume that (7) holds. Then $f = ag + \sum_{i=0}^{n} a_i \dfrac{\partial g}{\partial z_i}$, where $a, a_0, \ldots, a_n \in \mathcal{O}_{n+1}$. We may also assume that the order of g is $\geqq 2$, for otherwise, (ii)⇒(iii) is obvious. We will show that $a_i(0) = 0$, for $i = 0, \ldots, n$.

If not, we may find an invertible $(n+1) \times (n+1)$ matrix (α_{ij}) of complex numbers such that

$$\begin{aligned} \sum_i \alpha_{ij} a_i(0) &= 1, \quad \text{if } j = 0 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

We set $w_j = \sum_{i=0}^{n} \alpha_{ij} z_i$. Then $f = ag + \sum_{i=0}^{n} a_i \dfrac{\partial g}{\partial z_i} = ag + \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \alpha_{ij} a_i\right) \dfrac{\partial g}{\partial w_j}$. By using the coordinates $w_0, \ldots, w_n$ in place of $z_0, \ldots, z_n$ and $\sum_{j=0}^{n} \alpha_{ji} a_j$ in place of a_i, we may reduce the problem to the case when $a_0(0) = 1$ and $a_i(0) = 0$, for $i > 0$.

Let $U=\left\{\frac{\partial g}{\partial z_1}=\ldots=\frac{\partial g}{\partial z_n}=0\right\}$. Since U is defined by n equations in $n+1$ variables, and $U\cap\left\{\frac{\partial g}{\partial z_0}=0\right\}=0$, it follows that the germ of U at 0 is the germ of a curve. Let $\tau:(\mathbb{C},0)\to(U,0)$ be the normalization map-germ. Then $\frac{\partial g}{\partial z_i}\circ\tau=0$ for $i\geqq 1$ and $\frac{\partial g}{\partial z_0}\circ\tau\neq 0$.

Let $o(u)$ denote the order of u, when $u\in\mathcal{O}_1$. We have $\frac{d(g\circ\tau)}{dt}=\left(\frac{\partial g}{\partial z_0}\circ\tau\right)\frac{d\tau_0}{dt}$. Consequently, $o\left(\frac{\partial g}{\partial z_0}\circ\tau\right)<o(g\circ\tau)$. Since $f\circ\tau=(ag)\circ\tau+\left(a_0\frac{\partial g}{\partial z_0}\right)\circ\tau$, and $a_0(0)\neq 0$, we then obtain $o(f\circ\tau)=o\left(\frac{\partial g}{\partial z_0}\circ\tau\right)$.

Every ideal $J\subset\mathcal{O}_1$ is principal; we write $o(J)$ for the order of its generator. Since $\frac{d(f\circ\tau)}{dt}=\sum_{i=0}^{n}\left(\frac{\partial f}{\partial z_i}\circ\tau\right)\frac{d\tau_i}{dt}$, we obtain $o\left(\frac{\partial f}{\partial z_i}\circ\tau\right)<o(f\circ\tau)$, for at least one i. Hence

$$o\left(f\circ\tau,\frac{\partial f}{\partial z_i}\circ\tau\right)\mathcal{O}_1<o(f\circ\tau)=o\left(\frac{\partial g}{\partial z_0}\circ\tau\right)=o\left(g\circ\tau,\frac{\partial g}{\partial z_i}\circ\tau\right)\mathcal{O}_1.$$

But this contradicts (7).

We obtained this contradiction by assuming some $a_i(0)\neq 0$; therefore $a_i(0)=0$, for $i=0,\ldots,n$. Hence $f\in\left(g,z_i\frac{\partial g}{\partial z_j}\right)\mathcal{O}_{n+1}$. The same argument (with f and g interchanged) shows that $g\in\left(f,z_i\frac{\partial f}{\partial z_j}\right)\mathcal{O}_{n+1}$. From these facts and (7), we get (4). Hence (ii)$\Rightarrow$(iii). $\square$

§ 9. Related Literature

Dr. G.-M. Greuel has pointed out to us a paper by A.N. Shoshitaishvili [4] which studies the questions of right equivalence and right-left equivalence of holomorphic functions. To $f\in\mathcal{O}_{n+1}$, Shoshitaishvili associates the $\mathbb{C}$-algebra $Q(f)=\mathcal{O}_{n+1}\Big/\left(\frac{\partial f}{\partial z_0},\ldots,\frac{\partial f}{\partial z_n}\right)$. He considers the case $\dim_{\mathbb{C}}Q(f)<\infty$. Let $g\in\mathcal{O}_{n+1}$. When does $Q(g)\simeq Q(f)$ imply that g is right equivalent to f? Right-left equivalent? Shoshitaishvili gives necessary and sufficient conditions on f for each of these implications to hold. For right equivalence, one of his necessary and sufficient conditions is that f should be right equivalent to a quasi-homogeneous function. The other conditions are too technical to state here.

The equivalence for homogeneous functions was also proved independently by G.-M. Greuel and H.-P. Kraft and announced in a paper by Greuel [5], p. 231.

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