CLASSIFICATION OF GRADIENT SPACE AS $s\ell(2, \mathbb{C})$ MODULE I

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Dedicated to Professor Heisuke Hironaka on his sixtieth birthday.

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1. Introduction. Let M_n^k be the space of homogeneous polynomials of degree k in n variables $x_1, x_2, \ldots, x_n$. Let us fix a nontrivial $s\ell(2, \mathbb{C})$ action on M_n^1 (and hence on M_n^k). We shall denote S_n^k the subspace of M_n^k on which $s\ell(2, \mathbb{C})$ acts trivially. Let $S_n = \bigoplus_{k \geq 0} S_n^k$ be the graded ring of invariants. The main object of the invariant theory is to give explicit description of S_n in case $s\ell(2, \mathbb{C})$ acts on $\bigoplus_{k \geq 0} M_n^k$ via

$$\tau = (n-1)x_1\frac{\partial}{\partial x_1} + (n-3)x_2\frac{\partial}{\partial x_2} + \cdots + (-(n-3))x_{n-1}\frac{\partial}{\partial x_{n-1}} + (-(n-1))x_n\frac{\partial}{\partial x_n}$$

$$X_+ = (n-1)x_1\frac{\partial}{\partial x_2} + 2(n-2)x_2\frac{\partial}{\partial x_3} + \cdots + i(n-i)x_i\frac{\partial}{\partial x_{i+1}} + \cdots + (n-1)x_{n-1}\frac{\partial}{\partial x_n} \tag{1.1}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \cdots + x_i\frac{\partial}{\partial x_{i-1}} + \cdots + x_n\frac{\partial}{\partial x_{n-1}}.$$

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This situation is identical with the theory of binary quantics, which was diligently studied in second half of the nineteenth century. It is an amazingly difficult job to describe S_n explicitly. Complete success was achieved only for $n \leq 6$, the cases $n = 5$ and 6 being one of crowning glories of the theory. Elliott's book [El] has an excellent account on this subject. In 1967 Shioda [Sh] was able to describe S_8 explicitly.

In [Ya1] and [Ya2], the second author developed a new theory which connects isolated singularities on the one hand, and finite dimensional Lie algebras on the other hand. The natural question arising there is the following. Let f be a homogeneous polynomial of degree $k + 1$ in n variables. Consider the vector subspace $I(f)$ spanned by $\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$. Give a necessary and sufficient condition for $I(f)$ to be a $s\ell(2, \mathbb{C})$ submodule. If $I(f)$ is a $s\ell(2, \mathbb{C})$-submodule, give a complete classification of $I(f)$ as $s\ell(2, \mathbb{C})$-module. Here we consider all possible $s\ell(2, \mathbb{C})$ actions on $\mathbb{C}[[x_1, \ldots, x_n]]$ via derivations preserving the m-adic filtration. In [Ya4], the second author first observe that if $f \in S_n^{k+1}$ is an $s\ell(2, \mathbb{C})$ invariant polynomial, then $I(f)$ is a $s\ell(2, \mathbb{C})$-submodule. In this paper we shall only consider the $s\ell(2, \mathbb{C})$-action given by (1.1). In [Ya4], the second author proved that for $n \leq 5$, if $I(f)$ is a $s\ell(2, \mathbb{C})$ submodule, then $I(f) = (n)$ and f is an invariant polynomial, where (n) is an n-dimensional irreducible representation of $s\ell(2, \mathbb{C})$. The main purpose of this paper is to generalized this result.

Main Theorem. For $n \geq 2$, let f be a homogeneous polynomial of degree $k + 1 \geq 3$. If $I(f) = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is a $s\ell(2, \mathbb{C})$ submodule with respect to (1.1), then $I(f) = (n)$ and f is an invarient polynomial. Moreover $X_+ \, \partial f/\partial x_i = -i(n - i) \, \partial f/\partial x_{i+1}$, $X_- \, \partial f/\partial x_i = -\partial f/\partial x_{i-1}$, and $\tau \, \partial f/\partial x_i = -[n - (2i - 1)] \, \partial f/\partial x_i$ where we denote $\partial f/\partial x_0 = 0$ and $\partial f/\partial x_{n+1} = 0$.

In a subsequent paper we consider all possible reducible $s\ell(2, \mathbb{C})$ actions (i.e. all possible $s\ell(2, \mathbb{C})$ actions other than (1.1)). We are able to classify $I(f)$ as $s\ell(2, \mathbb{C})$ module. After completing the proof of the above results, the second author conjectured that the special case of our results can be generalized to other simple Lie algebras. This was finally proved by George Kempf, although his proof is somewhat complicated. The second author has applied the above results to prove the Lie algebras that he constructed from isolated hypersurface singularities (cf. [Ya1]) are solvable (cf. [Ya5]). This depends on the observation that the variety defined by $s\ell(2, \mathbb{C})$ invariant polynomial f is highly singular. As a consequence, the statement Theorem 1(a) of Kempf's paper [Ke] is vacuous. On the other hand, we do not know yet any application of his theorem other than the $s\ell(2, \mathbb{C})$ case. The proof of our main theorem is very elementary. We only make use of the classification theorem of $s\ell(2, \mathbb{C})$ representations which can be found for instance in Samelson's book [Sa]. Thus anyone can understand our proof easily.

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2. Notations and some lemmas. In this paper, we assume that $s\ell(2, \mathbb{C})$ acts on the space of homogeneous polynomials of degree $k + 1 \geq 3$ in $x_1, x_2, \ldots, x_n$, $n \geq 2$ by

$$\begin{aligned}\tau &= \sum_{i=1}^{n} [n - (2i - 1)]x_i \frac{\partial}{\partial x_i}\\ &= (n - 1)x_1 \frac{\partial}{\partial x_1} + (n - 3)x_2 \frac{\partial}{\partial x_2}\\ &\quad + \cdots + [-(n - 3)]x_{n-1} \frac{\partial}{\partial x_{n-1}} + [-(n - 1)]x_n \frac{\partial}{\partial x_n}\end{aligned}$$

$$\begin{aligned}X_+ &= \sum_{i=1}^{n-1} a_{i+1}x_i \frac{\partial}{\partial x_{i+1}}\\ &= a_2x_1 \frac{\partial}{\partial x_2} + a_3x_2 \frac{\partial}{\partial x_3} + \cdots + a_{n-1}x_{n-2} \frac{\partial}{\partial x_{n-1}} + a_nx_{n-1} \frac{\partial}{\partial x_n}\end{aligned}$$

where $a_2, \ldots, a_n$ are positive integers.

$$\begin{aligned}X_- &= \sum_{i=1}^{n-1} b_i x_{i+1} \frac{\partial}{\partial x_i}\\ &= b_1x_2 \frac{\partial}{\partial x_1} + b_2x_3 \frac{\partial}{\partial x_2} + \cdots + b_{n-2}x_{n-1} \frac{\partial}{\partial x_{n-2}} + b_{n-1}x_n \frac{\partial}{\partial x_{n-1}}\end{aligned}$$

where $b_1, \ldots, b_{n-1}$ are positive integers.

The weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.,

$$\mathrm{wt}(x_i) = n - (2i - 1) \qquad i = 1, 2, \ldots, n.$$

Assume $I = <\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n>$ is a $s\ell(2, \mathbb{C})$ module, where f is the homogeneous polynomial of degree $k + 1$. In the following we write $f = \sum_{j=-\infty}^{\infty} f_{k+1}^j$, where f_{k+1}^j is a homogeneous polynomial of degree $k + 1$ and weight j. If $I = (m)$, m-dimensional irreducible representation of $s\ell(2, \mathbb{C})$, then by the classification theorem of $s\ell(2, \mathbb{C})$, we know that $\partial f/\partial x_i$, $i = 1, 2, \ldots, n$, is a linear combination of homogeneous polynomials in I of degree k and weight $m - 1, m - 3, \ldots, -(m - 3), -(m - 1)$.

In what follows, if D_1 and D_2 are two differential operators, we shall denote $[D_1, D_2] = D_1D_2 - D_2D_1$ the commutator of D_1 and D_2.

LEMMA 2.1.

(a) *If* $g = \sum g^i \in I$, *where* g^i *is of weight* i, *then* $g^i \in I$, $\forall\, i$.
(b) $[\partial/\partial x_j, X_+] = a_{j+1}\, \partial/\partial x_{j+1}$, for $i \le j \le n$. *Here we denote* $a_{n+1} = 0$.
(c) *For any* i, $\partial/\partial x_j\, (X_+^\ell f_{k+1}^i) \in I$, *where* $i \le j \le n$ *and* $\ell \ge 0$.
(d) $[\partial/\partial x_j, X_-] = b_{j-1}\, \partial/\partial x_{j-1}$ *for* $i \le j \le n$. *Here we denote* $b_0 = 0$.
(e) *For any* i, $\partial/\partial x_j\, (X_-^\ell f_{k+1}^i) \in I$, *where* $i \le j \le n$, $\ell \ge 0$.

Proof. (a) Since $g \in I$ and I is a $s\ell(2, \mathbb{C})$-module, we have

$$\begin{aligned} g &= \sum g^i \in I \\ \tau(g) &= \sum i g^i \in I \\ \tau^2(g) &= \sum i^2 g^i \in I \\ &\vdots \\ \tau^n(g) &= \sum i^n g^i \in I. \end{aligned}$$

Because the Vandermonde matrix is invertible, we have $g^i \in I$, $\forall\, i$.

(b) and (d) These are immediate.

(c) We shall prove this by induction on ℓ. For $\ell = 0$, this follows from (a). Suppose that $\partial/\partial x_j\,(X_+^{\ell-1}f_{k+1}^i) \in I$ for any i and $1 \le j \le n$. By (b), we have the following equation.

$$\frac{\partial}{\partial x_j} X_+^{\ell} f_{k+l}^i = X_+ \frac{\partial}{\partial x_j}(X_+^{\ell-1} f_{k+1}^i) + a_{j+1}\frac{\partial}{\partial x_{j+1}}(X_+^{\ell-1} f_{k+1}^i).$$

Since I is a $s\ell(2, \mathbb{C})$-module, the right hand side of the above equation is in I by induction hypothesis.

(e) The proof is similar to that of (c). Q.E.D.

LEMMA 2.2.

(a) *If $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell$ depends only on x_1 variable, then $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell = 0$.*
(b) *If $X_+^i f_{k+1}^{j-2i}$ depends only on x_1 variable, then $X_+^i f_{k+1}^{j-2i} = 0$.*
(c) *If $\partial X_-^i f_{k+1}^{j+2i}/\partial x_\ell$ depends only on x_n variable, then $\partial X_-^i f_{k+1}^{j+2i}/\partial x_\ell = 0$.*
(d) *If $X_-^i f_{k+1}^{j+2i}$ depends only on x_n variable, then $X_-^i f_{k+1}^{j+2i} = 0$.*

Proof. (a) Since $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell \in I$ by lemma 1(c), $-(n-1) \le \mathrm{wt}(\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell) \le n-1$. Recall that $\mathrm{wt}(x_1) = n-1$. Therefore, if $\mathrm{wt}(\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell) < n-1$, then clearly $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell = 0$. Since $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell$ depends only on x_1, if $\mathrm{wt}(\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell) = n-1$, then $\partial X_+^i f_{k+1}^{j-2i}/\partial x_\ell = cx_1$ where c is a constant. As $k \ge 2$ by assumption, we have $c = 0$.

(b) If $X_+^i f_{k+1}^{j-2i}$ depends only on x_1, then so is $\partial X_+^i f_{k+1}^{j-2i}/\partial x_1$. By (a), $\partial X_+^i f_{k+1}^{j-2i}/\partial x_1 = 0$. This implies $X_+^i f_{k+1}^{j-2i} = 0$.

The proofs of (c) and (d) are similar to that of (a) and (b) respectively. Q.E.D.

LEMMA 2.3. *Let g be a homogeneous polynomial*

(a) *Suppose $X_+ g = 0$. If $\partial g/\partial x_\beta \ne 0$, then $\partial g/\partial x_j \ne 0$ for all $1 \le j \le \beta$.*
(b) *Suppose $X_- g = 0$. If $\partial g/\partial x_\beta \ne 0$, then $\partial g/\partial x_j \ne 0$ for all $\beta \le j \le n$.*

Proof.

$$0 = \frac{\partial}{\partial x_{\beta-1}} X_+ g = X_+ \frac{\partial g}{\partial x_{\beta-1}} + a_\beta \frac{\partial g}{\partial x_\beta}.$$

The above equation says that if $\partial g/\partial x_\beta \neq 0$, then $\partial g/\partial x_{\beta-1} \neq 0$. Statement (a) follows immediately by induction.

(b) Similarly, statement (b) follows from the following equation.

$$0 = \frac{\partial}{\partial x_{\beta+1}} X_- g = X_- \frac{\partial g}{\partial x_{\beta+1}} + b_\beta \frac{\partial g}{\partial x_\beta}.$$

Q.E.D.

LEMMA 2.4. *Let g be a homogeneous polynomial. Suppose $X_+ g = 0$.*

(a) *If $\partial g/\partial x_\beta = 0$, then $\partial g/\partial x_j = 0$ for all $\beta \leq j \leq n$.*
(b) *If $\partial^2 g/\partial x_\ell \partial x_\beta = 0$ and $\partial^2 g/\partial x_{\ell+1} \partial x_j = 0$, for all $\beta \leq j \leq n$, where $1 \leq \ell \leq n$, then $\partial^2 g/\partial x_\ell \partial x_j = 0$ for all $\beta \leq j \leq n$.*

Proof. (a) $0 = \partial/\partial x_\beta X_+ g = X_+ \partial g/\partial x_\beta + a_{\beta+1} \partial g/\partial x_{\beta+1}$. The above equation says that if $\partial g/\partial x_\beta = 0$, then $\partial g/\partial x_{\beta+1} = 0$. Statement (a) follows immediately by induction.

(b) Differentiate the above equation with respect to x_ℓ variable. We have

$$0 = X_+ \frac{\partial^2 g}{\partial x_\ell \partial x_\beta} + a_{\ell+1} \frac{\partial^2 g}{\partial x_{\ell+1} \partial x_\beta} + a_{\beta+1} \frac{\partial^2 g}{\partial x_\ell \partial x_{\beta+1}}.$$

The above equation says that if $\partial^2 g/\partial x_\ell \partial x_\beta = 0$ and $\partial^2 g/\partial x_{\ell+1} \partial x_\beta = 0$, then $\partial^2 g/\partial x_\ell \partial x_{\beta+1} = 0$. Statement (b) follows immediately by induction. Q.E.D.

3. Proof of the Main Theorem. We begin with special cases of the Main Theorem.

THEOREM 3.1. *Assume that $I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is a $s\ell(2, \mathbb{C})$-module, where f is a homogeneous polynomial of degree $k+1$, $k \geq 2$. If $I = (p)$* where *$p \leq n$, then f is a $s\ell(2, \mathbb{C})$ invariant polynomial and $I = (n)$. Moreover $X_+(\partial f/\partial x_i) = -a_{i+1} \partial f/\partial x_{i+1}$, $X_-(\partial f/\partial x_i) = -b_{i-1} \partial f/\partial x_{i-1}$ and $\tau(\partial f/\partial x_i) = -[n - (2i-1)] \partial f/\partial x_i$ where $1 \leq i \leq n$ and we denote $a_{n+1} = 0 = b_0$.*

Proof. Let $f = \sum_{j=-\infty}^{\infty} f^j$ where f^j is a homogeneous polynomial of degree $k+1$ and weight j. We shall prove by decreasing induction on j that $X_+^i f^{j-2i} = 0$ for all $j > 0$ and $i \geq 0$. Observe that for $j \geq 2(n-1)+1$

$$\operatorname{wt} \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} \geq 2(n-1) + 1 - (n-1) = n \qquad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = 0 \qquad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow X_+^i f^{j-2i} = 0.$$

Now suppose that $X_+^i (f^{j-2i}) = 0$ for all $i \geq 0$ and $j \geq m$. We are going to prove that $X_+^i (f^{m-1-2i}) = 0$ for all $i \geq 0$, provided $m > 1$.

Suppose that $X_+^i f^{m-1-2i}$ depends only on $x_1, x_2, \ldots, x_\alpha$. Since $m - 1 > 0$, $\operatorname{wt}(\partial X_+^i f^{m-1-2i}/\partial x_n) = m - 1 + (n-1) > n - 1$. Therefore, $\partial X_+^i f^{m-1-2i}/\partial x_n = 0$, i.e., $X_+^i f^{m-1-2i}$ is independent of x_n variable. Thus $1 \leq \alpha \leq n - 1$. We claim that $\partial X_+^i f^{m-1-2i}/\partial x_\alpha = 0$. For if $\partial X_+^i f^{m-1-2i}/\partial x_\alpha \neq 0$, then $\partial X_+^i f^{m-1-2i}/\partial x_j \neq 0$ for $1 \leq j \leq \alpha$ by lemma 2.3(a) since $X_+(X_+^i f^{m-1-2i}) = X_+^{i+1} f^{m+1-2(i+1)} = 0$ by induction hypothesis. Now for $2 \leq \ell \leq \alpha$, $\operatorname{wt}(X_- \partial X_+^i f^{m-1-2i}/\partial x_\ell) = m - n + 2\ell - 4 = \operatorname{wt}(\partial X_+^i f^{m-1-2i}/\partial x_{\ell-1})$. Since by hypothesis $I = (p)$, the vector subspace of I with weight $m - n + 2\ell - 4$ is of dimension one. There exists a constant c_ℓ such that

$$X_- \left(\frac{\partial X_+^i f^{m-1-2i}}{\partial x_\ell} \right) = c_\ell \frac{\partial X_+^i f^{m-1-2i}}{\partial x_{\ell-1}}.$$

Differentiate this equation with respect to the $x_{\alpha+1}$ variable, we have

$$X_- \frac{\partial^2 X_+^i f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_\ell} + b_\alpha \frac{\partial^2 X_+^i f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = c_\ell \frac{\partial^2 X_+^i f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_{\ell-1}}.$$

Since $X_+^i f^{m-1-2i}$ depends only on the variables $x_1, \ldots, x_\alpha$, the above equation implies

$$\frac{\partial^2 X_+^i f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = 0 \qquad \text{for } 2 \leq \ell \leq \alpha.$$

So $\partial X_+^i f^{m-1-2i}/\partial x_\alpha$ depends only the x_1 variable. In view of lemma 2(a), $\partial X_+^i f^{m-1-2i}/\partial x_\alpha = 0$. This simply means that $X_+^i f^{m-1-2i}$ is independent of x_α. By induction, we see that $X_+^i f^{m-1-2i}$ depends only on x_1. In view of lemma 2.2(b), we have $X_+^i f^{m-1-2i} = 0$. This completes our induction step. Hence we have shown

$$(*) \qquad X_+^i f^{j-2i} = 0 \qquad \text{for all } j > 0 \text{ and } i \geq 0$$

Similarly, we can prove

$$(**) \qquad X_-^i f^{j+2i} = 0 \qquad \text{for all } j < 0 \text{ and } i \geq 0$$

From (*) and (**), we conclude that $f^j = 0$ for $j \neq 0$. This means that f is the polynomial f^0 of weight 0. Notice that

$$X_+ f = X_+^1 f^0 = X_+^1 f^{2-2} = 0$$

by (*). Similarly,

$$X_- f = 0$$

by (**). In view of lemma 2.3, we know that if $f \neq 0$, then $\partial f/\partial x_i \neq 0$ for all $1 \leq i \leq n$. Since $\mathrm{wt}(\partial f/\partial x_i) = -\mathrm{wt}(x_i)$, $\partial f/\partial x_i, \ldots, \partial f/\partial x_n$ are linearly independent and hence $I = (n)$.

Observe that $0 = \partial/\partial x_i(X_+ f) = X_+(\partial f/\partial x_i) + a_{i+1}\, \partial f/\partial x_{i+1}$ for $i = 1, 2, \ldots, n-1$. So if we denote $a_{n+1} = 0$, then

$$X_+\left(\frac{\partial f}{\partial x_i}\right) = -a_{i+1}\frac{\partial f}{\partial x_{i+1}} \text{ for } i = 1, 2, \ldots, n$$

since

$$X_+\left(\frac{\partial f}{\partial x_n}\right) = \frac{\partial}{\partial x_n}(X_+ f) = 0.$$

Similarly,

$$0 = \frac{\partial}{\partial x_i}(X_- f) = X_- \frac{\partial f}{\partial x_i} + b_{i-1}\frac{\partial f}{\partial x_{i-1}} \text{ for } i = 2, 3, \ldots, n.$$

If we denote $b_0 = 0$, then

$$X_-\left(\frac{\partial f}{\partial x_i}\right) = -b_{i-1}\frac{\partial f}{\partial x_{i-1}} \text{ for } i = 1, 2, \ldots, n$$

since $X_-(\partial f/\partial x_1) = \partial/\partial x_1(X_- f) = 0$. Finally, $\tau(\partial f/\partial x_i) = -[n - (2i - 1)]\,\partial f/\partial x_i$ because $\mathrm{wt}(\partial f/\partial x_i) = -[n - (2i - 1)]$ Q.E.D.

PROPOSITION 3.2. *Assume that* $I = <\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_n>$ *is a* $s\ell(2, \mathbb{C})$*-module, where* f *is a homogeneous polynomial of degree* $k + 1$, $k \geq 2$. *Then* $I \neq (p_1) + (p_2) + \cdots + (p_q)$ *where* $n > p_1 \geq p_2 \cdots \geq p_q$, $q \geq 2$, $p_1 + p_2 + \cdots + p_q \leq n$ *and* $p_1 \leq [n/2]$ (*i.e.*, $n \geq 2p_1$).

Proof. Suppose $I = (p_1) + (p_2) + \cdots + (p_q)$. Let $f = \sum_{j=-\infty}^{\infty} f^j$ where f^j is a homogeneous polynomial of degree $k + 1$ and weight j. We shall prove by decreasing induction on j that $X_+^i f^{j-2i} = 0$ for all j, and for all $i \geq 0$. Observe that for $j \geq p_1 + n - 1$

$$\mathrm{wt}\frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1)$$

$$= p_1 \text{ for all } 1 \leq \ell \leq n$$

$$\Rightarrow \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = 0 \qquad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow X_+^i f^{j-2i} = 0.$$

Now suppose that $X_+^i f^{j-2i} = 0$ for all $i \geq 0$ and $j \geq m$. We are going to prove that $X_+^i(f^{m-1-2i}) = 0$ for all $i \geq 0$. Observe that

$$X_+\frac{\partial X_+^i f^{m-1-2i}}{\partial x_\ell} = \frac{\partial}{\partial x_\ell}X_+^{i+1}f^{m-1-2i} - a_{\ell+1}\frac{\partial}{\partial x_{\ell+1}}X_+^i f^{m-1-2i}$$

$$= \frac{\partial}{\partial x_\ell}X_+^{i+1}f^{m+1-2(i+1)} - a_{\ell+1}\frac{\partial}{\partial x_{\ell+1}}X_+^i f^{m-1-2i}$$

$$= -a_{\ell+1}\frac{\partial}{\partial x_{\ell+1}}X_+^i f^{m-1-2i}.$$

It follows that there are at most p_1 ℓ's such that

$$-(p_1 - 1) \leq \mathrm{wt}\,\frac{\partial X_+^i f^{m-1-2i}}{\partial x_\ell} \leq p_1 - 1,$$

so $X_+^i f^{m-1-2i}$ depends only on at most p_1 variables x_ℓ. In fact the above equation implies that $X_+^i f^{m-1-2i}$ depends only on $x_1, x_2, \ldots, x_\alpha$, where $1 \leq \alpha \leq p_1$. Hence $\partial X_+^i f^{m-1-2i}/\partial x_\ell$ depends only on the variables $x_1, x_2, \ldots, x_\alpha$, where $1 \leq \ell \leq \alpha$ and $\partial X_+^i f^{m-1-2i}/\partial x_\ell = 0$ for all $\alpha + 1 \leq \ell \leq n$. Since $\partial X_+^i f^{m-1-2i}/\partial x_\ell \in I$, so $\mathrm{wt}(\partial X_+^i f^{m-1-2i}/\partial x_\ell) \leq p_1 - 1 < n - 1$. Since $\mathrm{wt}(x_k) = n - 2k + 1 \geq n - 2\alpha + 1 \geq 2p_1 - 2p_1 + 1 = 1$ for all $1 \leq k \leq \alpha$ and $\mathrm{wt}(x_1) = n - 1$, so $\partial X_+^i f^{m-1-2i}/\partial x_\ell$ is independent of x_1 variable, i.e., $\partial^2 X_+^i f^{m-1-2i}/\partial x_\ell \partial x_1 = 0$. Now we have $\partial^2 X_+^i f^{m-1-2i}/\partial x_\ell \partial x_k = 0$ and $\partial^2 X_+^i f^{m-1-2i}/\partial x_k \partial x_1 = 0$ for $\alpha + 1 \leq \ell \leq n$ and $1 \leq k \leq n$. In particular, $\partial^2 X_+^i f^{m-1-2i}/\partial x_{\alpha+1} \partial x_k = 0$ and $\partial^2 X_+^i f^{m-1-2i}/\partial x_\alpha \partial x_1 = 0$ for $1 \leq k \leq n$. By lemma 2.4(b) $\partial^2 X_+^i f^{m-1-2i}/\partial x_\alpha \partial x_k = 0$ for all $1 \leq k \leq n$. By induction, we see that $\partial^2 X_+^i f^{m-1-2i}/\partial x_\ell \partial x_k = 0$ for all $1 \leq k \leq n$, $1 \leq \ell \leq n$. Thus $X_+^i f^{m-1-2i} = 0$. This completes our induction step. Therefore, $f^j = 0$ for all j. Thus $f = 0$ which contradicts $\deg f = k + 1 \geq 3$. Hence $I \neq (p_1) + (p_2) + \cdots + (p_q)$. Q.E.D.

Definition. If two integers are both odd or both even, they are said to have the same parity; if one is odd and the other even, they are said to have different parity. N integers are said to have the same parity if every two of them have the same parity, otherwise they are said to have different parity.

PROPOSITION 3.3. *Assume that $I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is a $s\ell(2, \mathbb{C})$-module, where f is a homogeneous polynomial of degree $k + 1$, $k \geq 2$. Then $I \neq (p_1) + (p_2) + \cdots + (p_q)$ where $n > p_1 \geq p_2 \geq \cdots \geq p_q$ and $q \geq 2$, $p_1 + p_2 + \cdots p_q \leq n$ and $p_1 > [n/2]$ (i.e., $n < 2p_1$) and $p_1, p_2, \cdots, p_q$ have the same parity.*

Proof. Suppose $I = (p_1) + (p_2) + \cdots + (p_q)$. Let $f = \sum_{j=-\infty}^{\infty} f^j$ where f^j is a homogeneous polynomial of degree $k + 1$ and weight j. We shall prove by decreasing induction on j that $X_+^i f^{j-2i} = 0$ for all j, and for all $i \geq 0$. Observe that for $j \geq p_1 + n - 1$

$$\mathrm{wt}\frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1) = p_1$$

for all $1 \leq \ell \leq n$

$$\Rightarrow \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = 0 \qquad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow X_+^i f^{j-2i} = 0.$$

Now suppose that $X_+^i f^{j-2i} = 0$ for all $j \geq m$. We are going to prove that $X_+^i f^{m-1-2i} = 0$. Observe that

$$X_+ \frac{\partial X_+^i f^{m-1-2i}}{\partial x_\ell} = \frac{\partial}{\partial x_\ell} X_+^{i+1} f^{m+1-2(i+1)} - a_{\ell+1} \frac{\partial}{\partial x_{\ell+1}} X_+^i f^{m-1-2i}$$

$$= -a_{\ell+1} \frac{\partial}{\partial x_{\ell+1}} X_+^i f^{m-1-2i}.$$

It follows that there are at most p_1 ℓ's such that

$$-(p_1 - 1) \leq \mathrm{wt} \frac{\partial X_+^i f^{m-1-2i}}{\partial x_\ell} \leq p_1 - 1,$$

so $X_+^i f^{m-1-2i}$ depends only on at most p_1 x_ℓ's variables. In fact the above equation implies that $X_+^i f^{m-1-2i}$ depends only on $x_1, x_2, \ldots, x_\alpha$ variables where $1 \leq \alpha \leq p_1$. If $\alpha \leq [n/2]$, then $X_+^i f^{m-1-2i} = 0$ by the argument of proposition 3.2. If $\alpha > [n/2]$ since $p_1, p_2, \ldots, p_q$ have the same parity, the possible weight of elements in I are $p_1 - 1, p_1 - 3, \ldots, -p_1 + 3, -p_1 + 1$. Since $p_1 > [n/2]$ and $p_1 + p_2 \leq n$, so $p_1 > p_2$ and $(p_1 - p_2)/2$ is a positive integer. Note that

$$\mathrm{wt} \frac{\partial X_+^i f^{m-1-2i}}{\partial x_\alpha} > \mathrm{wt} \frac{\partial X_+^i f^{m-1-2i}}{\partial x_{\alpha-1}} > \cdots > \mathrm{wt} \frac{\partial X_+^i f^{m-1-2i}}{\partial x_1}$$

and

$$\left\{\text{wt}\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\alpha}, \text{wt}\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\alpha-1}}, \ldots, \text{wt}\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_1}\right\}$$

$$\subseteq \{p_1 - 1, p_1 - 3, \ldots, p_2 + 1, p_2 - 1, p_2 - 3, \ldots,$$
$$-(p_2 - 3), -(p_2 - 1), -(p_2 + 1), \ldots,$$
$$-(p_1 - 3), -(p_1 - 1)\}.$$

Since the cardinal number of $\{p_1 - 1, p_1 - 3, \ldots, p_2 + 1, p_2 - 1, p_2 - 3, \ldots, -(p_2 - 3), -(p_2 - 1)\}$ is $(p_1 - p_2)/2 + p_2 = (p_1 + p_2)/2 \leq n/2 < \alpha$, there exists k with $1 \leq k < \alpha$ such that $\text{wt}\,\partial X^i_+ f^{m-1-2i}/\partial x_k = -(p_2 + 1)$. We now claim that $\partial X^i_+ f^{m-1-2i}/\partial x_\alpha = 0$. If $\partial X^i_+ f^{m-1-2i}/\partial x_\alpha \neq 0$, then $\partial X^i_+ f^{m-1-2i}/\partial x_\ell \neq 0$ for all $1 \leq \ell \leq \alpha$ by lemma 2.3(a). Note that the vector subspace of I with one of weight $\{p_1 - 1, p_1 - 3, \ldots, p_2 + 1, -(p_2 + 1), \ldots, -(p_1 - 3), -(p_1 - 1)\}$ is of dimension one. Now for $2 \leq \ell \leq k + 1$

$$\text{wt}\left(X_-\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\ell}\right)$$
$$= \text{wt}\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\ell-1}} \in \{-(p_2 + 1), \ldots, -(p_1 - 3), -(p_1 - 1)\}.$$

There exists a constant c_ℓ such that

$$X_-\left(\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\ell}\right) = c_\ell\,\frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\ell-1}}.$$

Differentiate this equation with respect to $x_{\alpha+1}$ variable, we have

$$b_\alpha\,\frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} + X_-\left(\frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1}\partial x_\ell}\right) = c_\ell\,\frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1}\partial x_{\ell-1}}.$$

Since $X_+^i f^{m-1-2i}$ depends only on $x_1, x_2, \ldots, x_\alpha$, the above equation implies

$$\frac{\partial^2 X_+^i f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = 0 \qquad \text{for all } 2 \leq \ell \leq k + 1.$$

In particular, $\partial^2 X_+^i f^{m-1-2i}/\partial x_\alpha \partial x_2 = 0$. Since $\partial^2 X_+^i f^{m-1-2i}/\partial x_{\alpha+1} \partial x_\ell = 0$ for $1 \leq \ell \leq n$, by lemma 2.4(b) $\partial^2 X_+^i f^{m-1-2i}/\partial x_\alpha \partial x_\ell = 0$ for all $2 \leq \ell \leq n$. Thus $\partial X_+^i f^{m-1-2i}/\partial x_\alpha$ depends only on x_1 variable. By lemma 2.2(a), $\partial X_+^i f^{m-1-2i}/\partial x_\alpha = 0$. This simply means that $X_+^i f^{m-1-2i}$ is independent of the x_α variable. By induction, we see that $X_+^i f^{m-1-2i}$ depends only on x_1. By lemma 2.2(b), $X_+^i f^{m-1-2i} = 0$. This completes our induction step. Therefore $f^j = 0$ for all j. Thus $f = 0$ and hence $I \neq (p_1) + (p_2) + \cdots + (p_q)$. Q.E.D.

THEOREM 3.4 *Assume that* $I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is *a* $s\ell(2, \mathbb{C})$*-module, where* f *is a homogeneous polynomial of degree* $k + 1$, $k \geq 2$. *Then* $I \neq (p_1) + (p_2) + \cdots + (p_q)$ *where* $n > p_1 \geq p_2 \geq \cdots \geq p_q$ *and* $q \geq 2$, $p_1 + p_2 + \cdots + p_q \leq n$.

Proof. If $p_1 \leq [n/2]$, then the theorem follows from proposition 3.2.

If $p_1 > [n/2]$ and $p_1, p_2, \ldots, p_q$ have the same parity, then the theorem follows from proposition 3.3.

If $p_1 > [n/2]$ and $p_1, p_2, \ldots, p_q$ have different parity, then we can divide $p_1, p_2, \ldots, p_q$ into two subsequences : $p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_s}$ and $p_{j_1} \geq p_{j_2} \geq \cdots p_{j_t}$, where $p_{i_1}, p_{i_2}, \ldots, p_{i_s}$ have the same parity and $p_{j_1}, p_{j_2}, \ldots, p_{j_t}$ have the same parity. Since $q \geq 2$, so $i_s \geq 1$ and $j_t \geq 1$.

Now suppose $I = (p_1) + (p_2) + \cdots + (p_q)$. Let $f = \sum_{j=-\infty}^{\infty} f^j$ where f^j is a homogeneous polynomial of degree $k + 1$ and weight j. We shall prove by decreasing induction on j that $X_+^i f^{j-2i} = 0$ for all j and for all $i \geq 0$. Observe that for $j \geq p_1 + n - 1$

$$\text{wt}\, \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1) = p_1$$

$$\text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow \frac{\partial X_+^i f^{j-2i}}{\partial x_\ell} = 0 \qquad \text{for all } 1 \le \ell \le n$$

$$\Rightarrow X_+^i f^{j-2i} = 0.$$

Now suppose that $X_+^i f^{j-2i} = 0$ for all $j \ge m$. We are going to prove that $X_+^i f^{m-1-2i} = 0$. Note that $\mathrm{wt}\,\partial X_+^i f^{m-1-2i}/\partial x_\ell$, $\ell = 1, 2, \ldots, n$ have the same parity. Suppose $\mathrm{wt}\,\partial X_+^i f^{m-1-2i}/\partial x_\ell \in \{p_{i_1} - 1, p_{i_1} - 3, \ldots, -(p_{i_1} - 3), -(p_{i_1} - 1)\}$. Consider the following two cases.

Case 1. If $i_S = 1$, since $p_{i_1} < n$, so $X_+^i f^{m-1-2i} = 0$ by the similar proof of Theorem 3.1.

Case 2. If $i_S \ge 2$ then $X_+^i f^{m-1-2i} = 0$ by the similar proof of proposition 3.2 or proposition 3.3 according to $p_{i_1} \le [n/2]$ or $p_{i_1} > [n/2]$.

Similarly, we can show that $X_+^i f^{m-1-2i} = 0$ if $\mathrm{wt}\,\partial X_+^i f^{m-1-2i}/\partial x_\ell \in \{p_{j_1} - 1, p_{j_1} - 3, \ldots, -(p_{j_1} - 3), -(p_{j_1} - 1)\}$. Thus in any case, we have $X_+^i f^{m-1-2i} = 0$. This completes our induction step. Therefore, $f^j = 0$ for all j. Thus $f = 0$ and hence $I \ne (p_1) + (p_2) + \cdots + (p_q)$. Q.E.D.

THEOREM 3.5. *For $n \ge 2$, let f be a homogeneous polynomial of degree $k + 1 \ge 3$. If $I(f) = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is a $s\ell(2, \mathbb{C})$ submodule with respect to* (1.1), *then $I(f) = (n)$ and f is an invariant polynomial. Moreover $X_+ \partial f/\partial x_i = -i(n-i)\partial f/\partial x_{i+1}$, $X_- \partial f/\partial x_i = -\partial f/\partial x_{i-1}$ and $\tau(\partial f/\partial x_i) = -[n - (2i-1)]\,\partial f/\partial x_i$ for $1 \le i \le n$ where $\partial f/\partial x_0 = 0 = \partial f/\partial x_{n+1}$.*

Proof. This follows immediately from Theorems 3.1 and Theorem 3.4. Q.E.D.

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