CLASSIFICATION OF GRADIENT SPACE AS $s\ell \left(2, \mathbb{C}\right)$

MODULE I

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Dedicated to Professor Heisuke Hironaka on his sixtieth birthday.

1. Introduction. Let $M_n^k$ be the space of homogeneous polynomials of degree $k$ in $n$ variables $x_1, x_2, \ldots, x_n$. Let us fix a nontrivial $s\ell \left(2, \mathbb{C}\right)$ action on $M_n^1$ (and hence on $M_n^k$). We shall denote $S_n^k$ the subspace of $M_n^k$ on which $s\ell \left(2, \mathbb{C}\right)$ acts trivially. Let $S_n = \bigoplus_{k \geq 0} S_n^k$ be the graded ring of invariants. The main object of the invariant theory is to give explicit description of $S_n$ in case $s\ell \left(2, \mathbb{C}\right)$ acts on $\bigoplus_{k \geq 0} M_n^k$ via

$$\tau = (n - 1)x_1 \frac{\partial}{\partial x_1} + (n - 3)x_2 \frac{\partial}{\partial x_2}$$

$$+ \cdots + \left(- (n - 3)\right)x_{n-1} \frac{\partial}{\partial x_{n-1}} + \left(- (n - 1)\right)x_n \frac{\partial}{\partial x_n}$$

(1.1) \hspace{1cm} \begin{align*}
X_+ &= (n - 1)x_1 \frac{\partial}{\partial x_2} + 2(n - 2)x_2 \frac{\partial}{\partial x_3} + \cdots + i(n - i)x_i \frac{\partial}{\partial x_{i+1}} \\
&\quad + \cdots + (n - 1)x_{n-1} \frac{\partial}{\partial x_n} \\
X_- &= x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \cdots + x_i \frac{\partial}{\partial x_{i-1}} + \cdots + x_n \frac{\partial}{\partial x_{n-1}}.
\end{align*}$$

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This situation is identical with the theory of binary quantics, which was
diligently studied in second half of the nineteenth century. It is an
amazingly difficult job to describe $S_n$ explicitly. Complete success was
achieved only for $n \leq 6$, the cases $n = 5$ and 6 being one of crowning
glories of the theory. Elliott's book [E1] has an excellent account on
this subject. In 1967 Shioda [Sh] was able to describe $S_8$ explicitly.

In [Ya1] and [Ya2], the second author developed a new theory
which connects isolated singularities on the one hand, and finite di-

dimensional Lie algebras on the other hand. The natural question arising
there is the following. Let $f$ be a homogeneous polynomial of degree
$k + 1$ in $n$ variables. Consider the vector subspace $I(f)$ spanned by
$\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$. Give a necessary and sufficient condition
for $I(f)$ to be a $\mathfrak{sl}(2, \mathbb{C})$ submodule. If $I(f)$ is a $\mathfrak{sl}(2, \mathbb{C})$-submodule,
give a complete classification of $I(f)$ as $\mathfrak{sl}(2, \mathbb{C})$-module. Here we con-
sider all possible $\mathfrak{sl}(2, \mathbb{C})$ actions on $\mathbb{C}[[x_1, \ldots, x_n]]$ via derivations
preserving the $m$-adic filtration. In [Ya4], the second author first observe
that if $f \in S_n^{k+1}$ is an $\mathfrak{sl}(2, \mathbb{C})$ invariant polynomial, then $I(f)$ is a
$\mathfrak{sl}(2, \mathbb{C})$-submodule. In this paper we shall only consider the $\mathfrak{sl}(2, \mathbb{C})$-
action given by (1.1). In [Ya4], the second author proved that for $n \leq 5$,
the Lie algebra $I(f)$ is a $\mathfrak{sl}(2, \mathbb{C})$ submodule, then $I(f) = (n)$ and $f$ is an invariant
polynomial, where $(n)$ is an $n$-dimensional irreducible representation of
$\mathfrak{sl}(2, \mathbb{C})$. The main purpose of this paper is to generalized this result.

**Main Theorem.** For $n \geq 2$, let $f$ be a homogeneous polynomial
of degree $k + 1 \geq 3$. If $I(f) = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle$ is a $\mathfrak{sl}(2, \mathbb{C})$
submodule with respect to (1.1), then $I(f) = (n)$ and $f$ is an invariant
polynomial. Moreover $X_+ \partial f/\partial x = -i(n - i) \partial f/\partial x_1, X_- \partial f/\partial x =
-\partial f/\partial x_i, \tau \partial f/\partial x = \gamma[n - (2i - 1)] \partial f/\partial x_i$, where we denote
$\partial f/\partial x_0 = 0$ and $\partial f/\partial x_{n+1} = 0$.

In a subsequent paper we consider all possible reducible $\mathfrak{sl}(2, \mathbb{C})$
actions (i.e. all possible $\mathfrak{sl}(2, \mathbb{C})$ actions other than (1.1)). We are able
to classify $I(f)$ as $\mathfrak{sl}(2, \mathbb{C})$ module. After completing the proof of the
above results, the second author conjectured that the special case of
our results can be generalized to other simple Lie algebras. This was
finally proved by George Kempf, although his proof is somewhat com-
plicated. The second author has applied the above results to prove the
Lie algebras that he constructed from isolated hypersurface singularities
(cf. [Ya1]) are solvable (cf. [Ya5]). This depends on the observation
that the variety defined by $\mathfrak{sl}(2, \mathbb{C})$ invariant polynomial $f$ is highly
singular. As a consequence, the statement Theorem 1(a) of Kempf's
paper [Ke] is vacuous. On the other hand, we do not know yet any application of his theorem other than the $s\ell(2, \mathbb{C})$ case. The proof of our main theorem is very elementary. We only make use of the classification theorem of $s\ell(2, \mathbb{C})$ representations which can be found for instance in Samelson’s book [Sa]. Thus anyone can understand our proof easily.

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2. Notations and some lemmas. In this paper, we assume that $s\ell(2, \mathbb{C})$ acts on the space of homogeneous polynomials of degree $k + 1 \geq 3$ in $x_1, x_2, \ldots, x_n$, $n \geq 2$ by

$$
\tau = \sum_{i=1}^{n} \left[n - (2i - 1)\right] x_i \frac{\partial}{\partial x_i}
\nonumber
\nonumber
\nonumber
$$

$$
= (n - 1)x_1 \frac{\partial}{\partial x_1} + (n - 3)x_2 \frac{\partial}{\partial x_2}
\nonumber
\nonumber
\nonumber
$$

$$
+ \cdots + [-(n - 3)] x_{n-1} \frac{\partial}{\partial x_{n-1}} + [-(n - 1)] x_n \frac{\partial}{\partial x_n}
\nonumber
\nonumber
\nonumber
$$

$$
X_+ = \sum_{i=1}^{n-1} a_{i+1} x_i \frac{\partial}{\partial x_{i+1}}
\nonumber
\nonumber
\nonumber
$$

$$
= a_2 x_1 \frac{\partial}{\partial x_2} + a_3 x_2 \frac{\partial}{\partial x_3} + \cdots + a_{n-1} x_{n-2} \frac{\partial}{\partial x_{n-1}} + a_n x_{n-1} \frac{\partial}{\partial x_n}
\nonumber
\nonumber
\nonumber
$$

where $a_2, \ldots, a_n$ are positive integers.

$$
X_- = \sum_{i=1}^{n-1} b_i x_{i+1} \frac{\partial}{\partial x_i}
\nonumber
\nonumber
\nonumber
$$

$$
= b_1 x_1 \frac{\partial}{\partial x_1} + b_2 x_2 \frac{\partial}{\partial x_2} + \cdots + b_{n-2} x_{n-1} \frac{\partial}{\partial x_{n-2}} + b_{n-1} x_{n-1} \frac{\partial}{\partial x_{n-1}}
\nonumber
\nonumber
\nonumber
$$
where \( b_1, \ldots, b_{n-1} \) are positive integers.

The weight of \( x \) is given by the corresponding coefficient in the expression of \( \tau \) above, i.e.,

\[
\text{wt}(x_i) = n - (2i - 1) \quad i = 1, 2, \ldots, n.
\]

Assume \( I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle \) is a \( \mathfrak{sl}(2, \mathbb{C}) \) module, where \( f \) is the homogeneous polynomial of degree \( k + 1 \). In the following we write \( f = \sum_{j=k+1}^n f_{k+1}^j \), where \( f_{k+1}^j \) is a homogeneous polynomial of degree \( k + 1 \) and weight \( j \). If \( I = (m) \), \( m \)-dimensional irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \), then by the classification theorem of \( \mathfrak{sl}(2, \mathbb{C}) \), we know that \( \partial f/\partial x_i, i = 1, 2, \ldots, n \), is a linear combination of homogeneous polynomials in \( I \) of degree \( k \) and weight \( m - 1, m - 3, \ldots, -(m - 3), -(m - 1) \).

In what follows, if \( D_1 \) and \( D_2 \) are two differential operators, we shall denote \([D_1, D_2] = D_1D_2 - D_2D_1\) the commutator of \( D_1 \) and \( D_2 \).

**Lemma 2.1.**

(a) If \( g = \sum g^i \in I \), where \( g^i \) is of weight \( i \), then \( g^i \in I, \forall i \).

(b) \([\partial/\partial x_i, X_+] = a_{i+1} \partial/\partial x_{i+1}, \quad \text{for} \quad i \leq j \leq n \). Here we denote \( a_{n+1} = 0 \).

(c) For any \( i, \partial/\partial x_i \) \( (X^\ell f_{k+1}) \in I \), where \( i \leq j \leq n \) and \( \ell \geq 0 \).

(d) \([\partial/\partial x_i, X_-] = b_{j-1} \partial/\partial x_{j-1}, \quad \text{for} \quad i \leq j \leq n \). Here we denote \( b_0 = 0 \).

(e) For any \( i, \partial/\partial x_i \) \( (X^\ell f_{k+1}) \in I \), where \( i \leq j \leq n, \ell \geq 0 \).

**Proof.** (a) Since \( g \in I \) and \( I \) is a \( \mathfrak{sl}(2, \mathbb{C}) \)-module, we have

\[
g = \sum g^i \in I
\]

\[
\tau(g) = \sum ig^i \in I
\]

\[
\tau^2(g) = \sum i^2g^i \in I
\]

\[
\vdots
\]

\[
\tau^n(g) = \sum i^ng^i \in I.
\]

Because the Vandermonde matrix is invertible, we have \( g^i \in I, \forall i \).

(b) and (d) These are immediate.

(c) We shall prove this by induction on \( \ell \). For \( \ell = 0 \), this follows
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from (a). Suppose that $\partial/\partial x_i (X_+^{i-1} f_{k+1}^{-1}) \in I$ for any $i$ and $1 \leq j \leq n$. By (b), we have the following equation.

$$\frac{\partial}{\partial x_j} X_+^i f_{k+1}^i = X_+ \frac{\partial}{\partial x_j} (X_+^{i-1} f_{k+1}^{-1}) + a_{i+1} \frac{\partial}{\partial x_{i+1}} (X_+^{i-1} f_{k+1}^{-1}).$$

Since $I$ is a $\mathfrak{s}(2, \mathbb{C})$-module, the right hand side of the above equation is in $I$ by induction hypothesis.

(c) The proof is similar to that of (c). Q.E.D.

Lemma 2.2.

(a) If $\partial X_+^i f_{k+1}^{-2i}/\partial x_i$ depends only on $x_1$ variable, then $\partial X_+^i f_{k+1}^{-2i}/\partial x_i = 0$.

(b) If $X_+^i f_{k+1}^{-2i}$ depends only on $x_1$ variable, then $X_+^i f_{k+1}^{-2i} = 0$.

(c) If $\partial X_-^i f_{k+1}^{+2i}/\partial x_i$ depends only on $x_n$ variable, then $\partial X_-^i f_{k+1}^{+2i}/\partial x_i = 0$.

(d) If $X_-^i f_{k+1}^{+2i}$ depends only on $x_n$ variable, then $X_-^i f_{k+1}^{+2i} = 0$.

Proof. (a) Since $\partial X_+^i f_{k+1}^{-2i}/\partial x_i \in I$ by lemma 1(c), $-(n - 1) \leq \text{wt}(\partial X_+^i f_{k+1}^{-2i}/\partial x_i) \leq n - 1$. Recall that $\text{wt}(x_1) = n - 1$. Therefore, if $\text{wt}(\partial X_+^i f_{k+1}^{-2i}/\partial x_i) < n - 1$, then clearly $\partial X_+^i f_{k+1}^{-2i}/\partial x_i = 0$. Since $\partial X_+^i f_{k+1}^{-2i}/\partial x_i$ depends only on $x_1$, if $\text{wt}(\partial X_+^i f_{k+1}^{-2i}/\partial x_i) = n - 1$, then $\partial X_+^i f_{k+1}^{-2i}/\partial x_i = cx_1$ where $c$ is a constant. As $k \geq 2$ by assumption, we have $c = 0$.

(b) If $X_+^i f_{k+1}^{-2i}$ depends only on $x_1$, then so is $\partial X_+^i f_{k+1}^{-2i}/\partial x_1$. By (a), $\partial X_+^i f_{k+1}^{-2i}/\partial x_1 = 0$. This implies $X_+^i f_{k+1}^{-2i} = 0$.

The proofs of (c) and (d) are similar to that of (a) and (b) respectively. Q.E.D.

Lemma 2.3. Let $g$ be a homogeneous polynomial

(a) Suppose $X_+ g = 0$. If $\partial g/\partial x_\beta \neq 0$, then $\partial g/\partial x_j \neq 0$ for all $1 \leq j \leq \beta$.

(b) Suppose $X_- g = 0$. If $\partial g/\partial x_\beta \neq 0$, then $\partial g/\partial x_j \neq 0$ for all $\beta \leq j \leq n$.

Proof.

$$0 = \frac{\partial}{\partial x_{\beta-1}} X_+ g = X_+ \frac{\partial g}{\partial x_{\beta-1}} + a_{\beta} \frac{\partial g}{\partial x_\beta}.$$
The above equation says that if \( \partial g/\partial x_\beta \neq 0 \), then \( \partial g/\partial x_{\beta-1} \neq 0 \). Statement (a) follows immediately by induction.

(b) Similarly, statement (b) follows from the following equation.

\[
0 = \frac{\partial}{\partial x_{\beta+1}} X_+ g = X_+ \frac{\partial g}{\partial x_{\beta+1}} + b_{\beta} \frac{\partial g}{\partial x_{\beta}}.
\]

Q.E.D.

**Lemma 2.4.** Let \( g \) be a homogeneous polynomial. Suppose \( X_+ g = 0 \).

(a) If \( \partial g/\partial x_\beta = 0 \), then \( \partial g/\partial x_j = 0 \) for all \( \beta \leq j \leq n \).

(b) If \( \partial^2 g/\partial x_\beta \partial x_\beta = 0 \) and \( \partial^2 g/\partial x_{\beta+1} \partial x_j = 0 \), for all \( \beta \leq j \leq n \), where \( 1 \leq \ell \leq n \), then \( \partial^2 g/\partial x_\beta \partial x_j = 0 \) for all \( \beta \leq j \leq n \).

**Proof.** (a) \( 0 = \partial/\partial x_\beta X_+ g = X_+ \partial g/\partial x_\beta + a_{\beta+1} \partial g/\partial x_{\beta+1} \). The above equation says that if \( \partial g/\partial x_\beta = 0 \), then \( \partial g/\partial x_{\beta+1} = 0 \). Statement (a) follows immediately by induction.

(b) Differentiate the above equation with respect to \( x_\ell \) variable. We have

\[
0 = X_+ \frac{\partial^2 g}{\partial x_\ell \partial x_\beta} + a_{\ell+1} \frac{\partial^2 g}{\partial x_{\ell+1} \partial x_\beta} + a_{\beta+1} \frac{\partial^2 g}{\partial x_\ell \partial x_{\beta+1}}.
\]

The above equation says that if \( \partial^2 g/\partial x_\beta \partial x_\beta = 0 \) and \( \partial^2 g/\partial x_{\beta+1} \partial x_\beta = 0 \), then \( \partial^2 g/\partial x_\ell \partial x_{\beta+1} = 0 \). Statement (b) follows immediately by induction.

Q.E.D.

3. **Proof of the Main Theorem.** We begin with special cases of the Main Theorem.

**Theorem 3.1.** Assume that \( I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle \) is a \( s\ell(2, \mathbb{C}) \)-module, where \( f \) is a homogeneous polynomial of degree \( k+1 \), \( k \geq 2 \). If \( I = (p) \) where \( p \leq n \), then \( f \) is a \( s\ell(2, \mathbb{C}) \) invariant polynomial and \( I = (n) \). Moreover \( X_+ (\partial f/\partial x_i) = -a_{i+1} \partial f/\partial x_{i+1}, X_- (\partial f/\partial x_i) = -b_{i-1} \partial f/\partial x_{i-1} \) and \( \tau(\partial f/\partial x_i) = -(n - (2i - 1)) \partial f/\partial x_i \), where \( 1 \leq i \leq n \) and we denote \( a_{n+1} = b_0 = 0 \).

**Proof.** Let \( f = \sum_{j=1}^{\infty} f^j \) where \( f^j \) is a homogeneous polynomial of degree \( k+1 \) and weight \( j \). We shall prove by decreasing induction
on $j$ that $X^i_+ f^{j-2i} = 0$ for all $j > 0$ and $i \geq 0$. Observe that for $j \geq 2 (n - 1) + 1$

$$\text{wt} \frac{\partial X^i_+ f^{j-2i}}{\partial x_i} \geq 2(n - 1) + 1 - (n - 1) = n \quad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow \frac{\partial X^i_+ f^{j-2i}}{\partial x_i} = 0 \quad \text{for all } 1 \leq \ell \leq n$$

$$\Rightarrow X^i_+ f^{j-2i} = 0.$$

Now suppose that $X^i_+ (f^{j-2i}) = 0$ for all $i \geq 0$ and $j \geq m$. We are going to prove that $X^i_+ (f^{m-1-2i}) = 0$ for all $i \geq 0$, provided $m > 1$.

Suppose that $X^i_+ f^{m-1-2i}$ depends only on $x_1, x_2, \ldots, x_n$. Since $m - 1 > 0$, $\text{wt}(\partial X^i_+ f^{m-1-2i}/\partial x_n) = m - 1 + (n - 1) > n - 1$. Therefore, $\partial X^i_+ f^{m-1-2i}/\partial x_n = 0$, i.e., $X^i_+ f^{m-1-2i}$ is independent of $x_1$ variable. Thus $1 \leq \alpha \leq n - 1$. We claim that $\partial X^i_+ f^{m-1-2i}/\partial x_\alpha = 0$. For if $\partial X^i_+ f^{m-1-2i}/\partial x_\alpha \neq 0$, then $\partial X^i_+ f^{m-1-2i}/\partial x_j \neq 0$ for $1 \leq j \leq \alpha$ by lemma 2.3(a) since $X_+ (X^i_+ f^{m-1-2i}) = X^i_+ f^{m+1-2i}/\partial x_{\alpha+1} = 0$ by induction hypothesis. Now for $2 \leq \ell \leq \alpha$, $\text{wt}(X_+ \partial X^i_+ f^{m-1-2i}/\partial x_\ell) = m - n + 2\ell - 4 = \text{wt}(\partial X^i_+ f^{m-1-2i}/\partial x_{\ell-1})$. Since by hypothesis $I = (p)$, the vector subspace of $I$ with weight $m - n + 2\ell - 4$ is of dimension one. There exists a constant $c_\ell$ such that

$$X_+ \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\ell} = c_\ell \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\ell-1}}.$$

Differentiate this equation with respect to the $x_{\alpha+1}$ variable, we have

$$X_+ \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_\ell} + b_\alpha \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = c_\ell \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_{\ell-1}}.$$

Since $X^i_+ f^{m-1-2i}$ depends only on the variables $x_1, \ldots, x_\alpha$, the above equation implies

$$\frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = 0 \quad \text{for } 2 \leq \ell \leq \alpha.$$
So \( \partial X^+_+ f^{m-1-2i}/\partial x_\alpha \) depends only the \( x_i \) variable. In view of lemma 2(a), \( \partial X^+_+ f^{m-1-2i}/\partial x_\alpha = 0 \). This simply means that \( X_+ f^{m-1-2i} \) is independent of \( x_\alpha \). By induction, we see that \( X^+_+ f^{m-1-2i} \) depends only on \( x_1 \). In view of lemma 2.2(b), we have \( X^+_+ f^{m-1-2i} = 0 \). This completes our induction step. Hence we have shown

\[
(*) \quad X^+_+ f^{i-2} = 0 \quad \text{for all } j > 0 \text{ and } i \geq 0
\]

Similarly, we can prove

\[
(**) \quad X^- f^{i+2} = 0 \quad \text{for all } j < 0 \text{ and } i \geq 0
\]

From (*) and (**), we conclude that \( f^j = 0 \) for \( j \neq 0 \). This means that \( f \) is the polynomial \( f^0 \) of weight 0. Notice that

\[
X^+_+ f = X^+_+ f^0 = X^+_+ f^{2-2} = 0
\]

by (*). Similarly,

\[
X^- f = 0
\]

by (**). In view of lemma 2.3, we know that if \( f \neq 0 \), then \( \partial f/\partial x_i \neq 0 \) for all \( 1 \leq i \leq n \). Since \( \text{wt}(\partial f/\partial x_i) = -\text{wt}(x_i) \), \( \partial f/\partial x_i, \ldots, \partial f/\partial x_n \) are linearly independent and hence \( l = (n) \).

Observe that \( 0 = \partial/\partial x_i(X^+_+ f) = X^+_+(\partial f/\partial x_i) + a_{i+1} \partial f/\partial x_{i+1} \) for \( i = 1, 2, \ldots, n-1 \). So if we denote \( a_{n+1} = 0 \), then

\[
X^+_+(\partial f/\partial x_i) = -a_{i+1} \frac{\partial f}{\partial x_{i+1}} \quad \text{for } i = 1, 2, \ldots, n
\]

since

\[
X^+_+(\partial f/\partial x_n) = \frac{\partial}{\partial x_n}(X^+_+ f) = 0.
\]

Similarly,

\[
0 = \frac{\partial}{\partial x_i}(X^- f) = X^- \frac{\partial f}{\partial x_i} + b_{i-1} \frac{\partial f}{\partial x_{i-1}} \quad \text{for } i = 2, 3, \ldots, n.
\]
If we denote \( b_0 = 0 \), then

\[
X_-(\frac{\partial f}{\partial x_i}) = -b_{i-1} \frac{\partial f}{\partial x_{i-1}} \text{ for } i = 1, 2, \ldots, n
\]

since \( X_-(\frac{\partial f}{\partial x_1}) = \partial/\partial x_1(X_-f) = 0 \). Finally, \( \tau(\frac{\partial f}{\partial x_i}) = -[n - (2i - 1)] \frac{\partial f}{\partial x_i} \), because \( \text{wt}(\frac{\partial f}{\partial x_i}) = -[n - (2i - 1)] \)

Q.E.D.

**Proposition 3.2.** Assume that \( I = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \rangle \) is a \( sl(2, \mathbb{C}) \)-module, where \( f \) is a homogeneous polynomial of degree \( k + 1, k \geq 2 \). Then \( I \neq (p_1) + (p_2) + \cdots + (p_n) \) where \( n > p_1 \geq p_2 \cdots \geq p_q, q \geq 2, p_1 + p_2 + \cdots + p_q \leq n \) and \( p_1 \leq [n/2] \) (i.e., \( n \geq 2p_1 \)).

**Proof.** Suppose \( I = (p_1) + (p_2) + \cdots + (p_q) \). Let \( f = \Sigma_{j=m}^{\infty} f^j \) where \( f^j \) is a homogeneous polynomial of degree \( k + 1 \) and weight \( j \). We shall prove by decreasing induction on \( j \) that \( X_+ f^{j-2i} = 0 \) for all \( j \), and for all \( i \geq 0 \). Observe that for \( j \geq p_1 + n - 1 \)

\[
\text{wt}\frac{\partial X^j f^{j-2i}}{\partial x_\ell} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1) = p_1 \text{ for all } 1 \leq \ell \leq n
\]

\[
\Rightarrow \frac{\partial X^j f^{j-2i}}{\partial x_\ell} = 0 \text{ for all } 1 \leq \ell \leq n
\]

\[
\Rightarrow X_+ f^{j-2i} = 0.
\]

Now suppose that \( X_+ f^{j-2i} = 0 \) for all \( i \geq 0 \) and \( j \geq m \). We are going to prove that \( X^i (f^{m-1-2i}) = 0 \) for all \( i \geq 0 \). Observe that

\[
X_+ \frac{\partial X^i f^{m-1-2i}}{\partial x_\ell} = \frac{\partial}{\partial x_\ell} X^i f^{m-1-2i} - a_{\ell+1} \frac{\partial}{\partial x_{\ell+1}} X^i f^{m-1-2i}
\]

\[
= \frac{\partial}{\partial x_\ell} X^i f^{m+1-2(i+1)} - a_{\ell+1} \frac{\partial}{\partial x_{\ell+1}} X^i f^{m-1-2i}
\]

\[
= -a_{\ell+1} \frac{\partial}{\partial x_{\ell+1}} X^i f^{m-1-2i}.
\]
It follows that there are at most \( p_1 \ell \)'s such that

\[-(p_1 - 1) \leq \text{wt} \left\{ \frac{\partial X_i^i f^{m-1-2i}}{\partial x_i} \right\} \leq p_1 - 1,
\]

so \( X_i f^{m-1-2i} \) depends only on at most \( p_1 \) variables \( x_i \). In fact the above equation implies that \( X_i^i f^{m-1-2i} \) depends only on \( x_1, x_2, \ldots, x_a \), where \( 1 \leq \alpha \leq p_1 \). Hence \( \partial X_i^i f^{m-1-2i}/\partial x_i \) depends only on the variables \( x_1, x_2, \ldots, x_a \), where \( 1 \leq \ell \leq \alpha \) and \( \partial X_i^i f^{m-1-2i}/\partial x_i = 0 \) for all \( \alpha + 1 \leq \ell \leq n \). Since \( \partial X_i^i f^{m-1-2i}/\partial x_i \in I \), so \( \text{wt}(\partial X_i^i f^{m-1-2i}/\partial x_i) \leq p_1 - 1 < n - 1 \). Since \( \text{wt}(x_i) = n - 2k + 1 \geq n - 2\alpha + 1 \geq 2p_1 - 2p_1 + 1 = 1 \) for all \( 1 \leq k \leq \alpha \) and \( \text{wt}(x_i) = n - 1 \), so \( \partial X_i^i f^{m-1-2i}/\partial x_i \) is independent of \( x_i \) variables, i.e., \( \partial^2 X_i^i f^{m-1-2i}/\partial x_1 = 0 \). Now we have \( \partial^2 X_i^i f^{m-1-2i}/\partial x_k = 0 \) and \( \partial^2 X_i^i f^{m-1-2i}/\partial x_k \cdot \partial x_1 = 0 \) for \( \alpha + 1 \leq \ell \leq n \) and \( 1 \leq k \leq n \). In particular, \( \partial^2 X_i^i f^{m-1-2i}/\partial x_1 = 0 \). By lemma 2.4(b), \( \partial^2 X_i^i f^{m-1-2i}/\partial x_1 = 0 \) for all \( 1 \leq k \leq n \). By induction, we see that \( \partial^2 X_i^i f^{m-1-2i}/\partial x_k = 0 \) for all \( 1 \leq k \leq n \), \( 1 \leq \ell \leq n \). Thus \( X_i^i f^{m-1-2i} = 0 \). This completes our induction step. Therefore, \( f = 0 \) for all \( j \). Thus \( f = 0 \) which contradicts \( \text{deg} f = k + 1 \geq 3 \). Hence \( I \neq (p_1) + (p_2) + \cdots + (p_a) \). Q.E.D.

**Definition.** If two integers are both odd or both even, they are said to have the same parity; if one is odd and the other even, they are said to have different parity. \( N \) integers are said to have the same parity if every two of them have the same parity, otherwise they are said to have different parity.

**Proposition 3.3.** Assume that \( I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle \) is a \( s\ell(2, \mathbb{C}) \)-module, where \( f \) is a homogeneous polynomial of degree \( k + 1 \), \( k \geq 2 \). Then \( I \neq (p_1) + (p_2) + \cdots + (p_a) \) where \( n > p_1 \geq p_2 \geq \cdots \geq p_a \) and \( q \geq 2, p_1 + p_2 + \cdots + p_a \leq n \) and \( p_1 > [n/2] \) (i.e., \( n < 2p_1 \)) and \( p_1, p_2, \ldots, p_a \) have the same parity.

**Proof.** Suppose \( I = (p_1) + (p_2) + \cdots + (p_a) \). Let \( f = \Sigma f^i \) where \( f^i \) is a homogeneous polynomial of degree \( k + 1 \) and weight \( j \). We shall prove by decreasing induction on \( j \) that \( X_i^i f^{j-2i} = 0 \) for all \( j \), and for all \( i \geq 0 \). Observe that for \( j \geq p_1 + n - 1 \)
\[ \text{wt} \frac{\partial X^i f^{m-1-2i}}{\partial x_t} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1) = p_1 \]

for all \( 1 \leq \ell \leq n \)

\[ \Rightarrow \frac{\partial X^i f^{m-1-2i}}{\partial x_t} = 0 \quad \text{for all} \quad 1 \leq \ell \leq n \]

\[ \Rightarrow X^i f^{m-1-2i} = 0. \]

Now suppose that \( X^i f^{m-1-2i} = 0 \) for all \( j \geq m \). We are going to prove that \( X^i f^{m-1-2i} = 0 \). Observe that

\[
X^i \frac{\partial X^i f^{m-1-2i}}{\partial x_t} = \frac{\partial}{\partial x_t} X^i f^{m+1-2(i+1)} - a_{t+1} \frac{\partial}{\partial x_{t+1}} X^i f^{m-1-2i}
\]

\[
= -a_{t+1} \frac{\partial}{\partial x_{t+1}} X^i f^{m-1-2i}.
\]

It follows that there are at most \( p_1 \ell \)'s such that

\[-(p_1 - 1) \leq \text{wt} \frac{\partial X^i f^{m-1-2i}}{\partial x_t} \leq p_1 - 1,\]

so \( X^i f^{m-1-2i} \) depends only on at most \( p_1 \) \( x_t \)'s variables. In fact the above equation implies that \( X^i f^{m-1-2i} \) depends only on \( x_1, x_2, \ldots, x_\alpha \) variables where \( 1 \leq \alpha \leq p_1 \). If \( \alpha \leq \lfloor n/2 \rfloor \), then \( X^i f^{m-1-2i} = 0 \) by the argument of proposition 3.2. If \( \alpha > \lfloor n/2 \rfloor \) since \( p_1, p_2, \ldots, p_q \) have the same parity, the possible weight of elements in \( I \) are \( p_1 - 1, p_1 - 3, \ldots, -p_1 + 3, -p_1 + 1 \). Since \( p_1 > \lfloor n/2 \rfloor \) and \( p_1 + p_2 \leq n \), so \( p_1 > p_2 \) and \( (p_1 - p_2)/2 \) is a positive integer. Note that

\[
\text{wt} \frac{\partial X^i f^{m-1-2i}}{\partial x_\alpha} > \text{wt} \frac{\partial X^i f^{m-1-2i}}{\partial x_{\alpha-1}} > \cdots > \text{wt} \frac{\partial X^i f^{m-1-2i}}{\partial x_1}
\]
and

\[
\left\{ \text{wt} \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\alpha}}, \text{wt} \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\alpha-1}}, \ldots, \text{wt} \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{1}} \right\}
\]

\[
\subseteq \{ p_1 - 1, p_1 - 3, \ldots, p_2 + 1, p_2 - 1, p_2 - 3, \ldots, -(p_2 - 3), -(p_2 - 1), -(p_2 + 1), \ldots, -(p_1 - 3), -(p_1 - 1) \}.
\]

Since the cardinal number of \{p_1 - 1, p_1 - 3, \ldots, p_2 + 1, p_2 - 1, p_2 - 3, \ldots, -(p_2 - 3), -(p_2 - 1)\} is \((p_1 - p_2)/2 + p_2 = (p_1 + p_2)/2 \leq n/2 < \alpha\), there exists \(k\) with \(1 \leq k < \alpha\) such that \(\text{wt} \partial X^i_+ f^{m-1-2i}/\partial x_k = -(p_2 + 1)\). We now claim that \(\partial X^i_+ f^{m-1-2i}/\partial x_a = 0\). If \(\partial X^i_a f^{m-1-2i}/\partial x_a \neq 0\), then \(\partial X^i_+ f^{m-1-2i}/\partial x_\ell \neq 0\) for all \(1 \leq \ell \leq \alpha\) by lemma 2.3(a). Note that the vector subspace of \(I\) with one of weight \{p_1 - 1, p_1 - 3, \ldots, p_2 + 1, -(p_2 + 1), \ldots, -(p_1 - 3), -(p_1 - 1)\} is of dimension one. Now for \(2 \leq \ell \leq k + 1\)

\[
\text{wt} \left( X_- \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\ell} \right) = \text{wt} \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\ell-1}} \in \{ -(p_2 + 1), \ldots, -(p_1 - 3), -(p_1 - 1) \}.
\]

There exists a constant \(c_\ell\) such that

\[
X_- \left( \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_\ell} \right) = c_\ell \frac{\partial X^i_+ f^{m-1-2i}}{\partial x_{\ell-1}}.
\]

Differentiate this equation with respect to \(x_{\alpha+1}\) variable, we have

\[
b_\alpha \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} + X_- \left( \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_\ell} \right) = c_\ell \frac{\partial^2 X^i_+ f^{m-1-2i}}{\partial x_{\alpha+1} \partial x_{\ell-1}}.
\]
Since \( X^i, f^{m-1-2i} \) depends only on \( x_1, x_2, \ldots, x_\alpha \), the above equation implies

\[
\frac{\partial^2 X^i, f^{m-1-2i}}{\partial x_\alpha \partial x_\ell} = 0 \quad \text{for all } 2 \leq \ell \leq k + 1.
\]

In particular, \( \partial^2 X^i, f^{m-1-2i}/\partial x_\alpha \partial x_2 = 0 \). Since \( \partial^2 X^i, f^{m-1-2i}/\partial x_\alpha + 1 \partial x_\ell = 0 \) for \( 1 \leq \ell \leq n \), by lemma 2.4(b) \( \partial^2 X^i, f^{m-1-2i}/\partial x_\alpha \partial x_\ell = 0 \) for all \( 2 \leq \ell \leq n \). Thus \( \partial X^i, f^{m-1-2i}/\partial x_\alpha \) depends only on \( x_1 \) variable. By lemma 2.2(a), \( \partial X^i, f^{m-1-2i}/\partial x_\alpha = 0 \). This simply means that \( X^i, f^{m-1-2i} \) is independent of the \( x_\alpha \) variable. By induction, we see that \( X^i, f^{m-1-2i} \) depends only on \( x_1 \). By lemma 2.2(b), \( X^i, f^{m-1-2i} = 0 \). This completes our induction step. Therefore \( f^i = 0 \) for all \( j \). Thus \( f = 0 \) and hence \( I \neq (p_1) + (p_2) + \cdots + (p_\alpha) \). Q.E.D.

**Theorem 3.4** Assume that \( I = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle \) is a \( sl(2, \mathbb{C}) \)-module, where \( f \) is a homogeneous polynomial of degree \( k + 1, k \geq 2 \). Then \( I \neq (p_1) + (p_2) + \cdots + (p_\alpha) \) where \( n > p_1 \geq p_2 \geq \cdots \geq p_\alpha \) and \( q \geq 2, p_1 + p_2 + \cdots + p_\alpha \leq n \).

**Proof.** If \( p_1 \leq [n/2] \), then the theorem follows from proposition 3.2.

If \( p_1 > [n/2] \) and \( p_1, p_2, \ldots, p_\alpha \) have the same parity, then the theorem follows from proposition 3.3.

If \( p_1 > [n/2] \) and \( p_1, p_2, \ldots, p_\alpha \) have different parity, then we can divide \( p_1, p_2, \ldots, p_\alpha \) into two subsequences : \( p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_\alpha} \) and \( p_{j_1} \geq p_{j_2} \geq \cdots \geq p_{j_\alpha} \), where \( p_{i_1}, p_{i_2}, \ldots, p_{i_\alpha} \) have the same parity and \( p_{j_1}, p_{j_2}, \ldots, p_{j_\alpha} \) have the same parity. Since \( q \geq 2 \), so \( i_\alpha \geq 1 \) and \( j_\alpha \geq 1 \).

Now suppose \( I = (p_1) + (p_2) + \cdots + (p_\alpha) \). Let \( f = \Sigma_{j=1}^{\alpha} f^i \) where \( f^i \) is a homogeneous polynomial of degree \( k + 1 \) and weight \( j \). We shall prove by decreasing induction on \( j \) that \( X^i, f^{j-2i} = 0 \) for all \( j \) and for all \( i \geq 0 \). Observe that for \( j \geq p_1 + n - 1 \)

\[
\text{wt} \frac{\partial X^i, f^{j-2i}}{\partial x_\ell} = j - [n - (2\ell - 1)] \geq p_1 + n - 1 - (n - 1) = p_1
\]

for all \( 1 \leq \ell \leq n \).
\[ \frac{\partial X_i^\ell f^{j-2i}}{\partial x_\ell} = 0 \quad \text{for all } 1 \leq \ell \leq n \]

\[ X_i^\ell f^{j-2i} = 0. \]

Now suppose that \( X_i^\ell f^{j-2i} = 0 \) for all \( j \geq m \). We are going to prove that \( X_i f^{n-1-2i} = 0 \). Note that \( \text{wt} \partial X_i^\ell f^{m-1-2i}/\partial x_\ell, \ell = 1, 2, \ldots, n \) have the same parity. Suppose \( \text{wt} \partial X_i^\ell f^{m-1-2i}/\partial x_\ell \in \{p_i - 1, p_i - 3, \ldots, -(p_i - 3), -(p_i - 1)\} \). Consider the following two cases.

**Case 1.** If \( i_S = 1 \), since \( p_i < n \), so \( X_i f^{m-1-2i} = 0 \) by the similar proof of Theorem 3.1.

**Case 2.** If \( i_S \geq 2 \) then \( X_i f^{m-1-2i} = 0 \) by the similar proof of proposition 3.2 or proposition 3.3 according to \( p_i \leq \lceil n/2 \rceil \) or \( p_i > \lceil n/2 \rceil \).

Similarly, we can show that \( X_i f^{m-1-2i} = 0 \) if \( \text{wt} \partial X_i f^{m-1-2i}/\partial x_\ell \in \{p_i - 1, p_i - 3, \ldots, -(p_i - 3), -(p_i - 1)\} \). Thus in any case, we have \( X_i f^{m-1-2i} = 0 \). This completes our induction step. Therefore, \( f^j = 0 \) for all \( j \). Thus \( f = 0 \) and hence \( I \neq (p_1) + (p_2) + \cdots + (p_a) \).

**Q.E.D.**

**Theorem 3.5.** For \( n \geq 2 \), let \( f \) be a homogeneous polynomial of degree \( k + 1 \geq 3 \). If \( I(f) = \langle \partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n \rangle \) is a \( \mathfrak{sl}(2, \mathbb{C}) \) submodule with respect to (1.1), then \( I(f) = (n) \) and \( f \) is an invariant polynomial. Moreover \( X_i \partial f/\partial x_i = -i(n - i) \partial f/\partial x_i, X_{i-1} \partial f/\partial x_i = -\partial f/\partial x_{i-1} \) and \( \tau(\partial f/\partial x_i) = -[n - (2i - 1)] \partial f/\partial x_i \) for \( 1 \leq i \leq n \) where \( \partial f/\partial x_0 = 0 = \partial f/\partial x_{n+1} \).

**Proof.** This follows immediately from Theorems 3.1 and Theorem 3.4.

Q.E.D.

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REFERENCES


